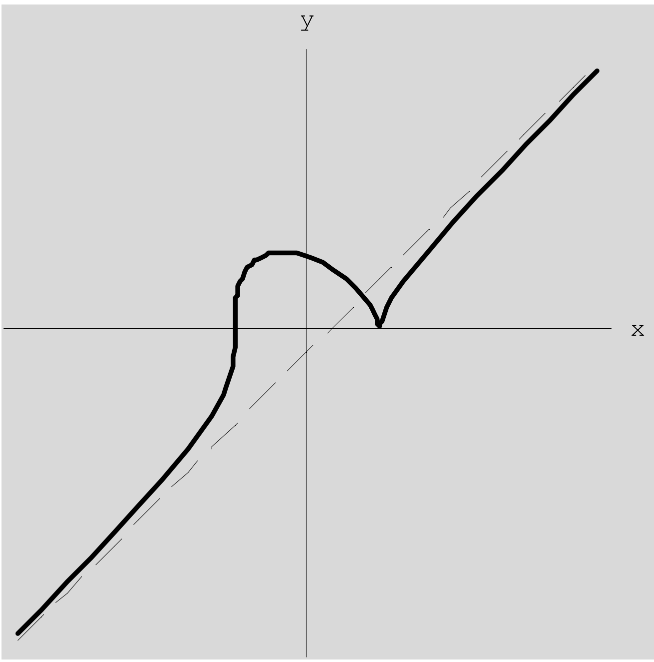
SVEUČILIŠTE U ZAGREBU
GEODETSKI FAKULTET

MATEMATIKA I

ZBIRKA RIJEŠENIH ZADATAKA S PISMENIH ISPITA

Odabrala i pripremila:
Jelena Beban-Brkić



Zagreb, 2001.

Ispitna zadaća 1:

1. U skupu kompleksnih brojeva riješiti jednadžbu $(2+i)z^2-(5-i)z+2-2i=0$.

2. Dana je funkcija $g(x)=arc\,cos\left(sin\,3^{\sqrt[3]{-x}}\right)$. Izračunati $g'(-1)$.

3. Analizirati funkciju $f(x)=\frac{x^3+2x^2+7x-3}{2x^2}$ i nacrtaj njen graf.

4. Vektor $\vec{c}$ u bazi $(\vec{i},\vec{j},\vec{k})$ ima komponente (16, -15, 12). Odrediti koordinate vektora $\vec{d}$ ako je $\vec{d}$ koolinearan s $\vec{c}$, suprotno orijentiran i ako je $|\vec{d}|=75$.

5. Izračunati približno $cos(299°)$ pomoću diferencijala.

6. Odrediti projekciju točke $T(1,2,8)$ na pravac $p\ldots\begin{cases}x+2y-1=0\\ y+z=0\end{cases}$.

Rješenja:

1. U skupu kompleksnih brojeva riješiti jednadžbu $(2+i)z^2-(5-i)z+2-2i=0$.

Rješenje:

$$z_{1.2}=\frac{(5-i)\pm\sqrt{(5-i)^2-4(2+i)(2-2i)}}{2(2+i)}=\frac{(5-i)\pm\sqrt{-2i}}{2(2+i)}$$

$$\omega=-2i,\quad \sqrt{-2i}=?$$

$$\Rightarrow|\omega|=2,\quad \varphi=\frac{3\pi}{2},\quad \omega=2\left(cos\frac{3\pi}{2}+i\,sin\frac{3\pi}{2}\right)$$

$$\Rightarrow\left(\sqrt{\varpi}\right)_k=\sqrt{|\omega|}\left(cos\frac{\varphi+2k\pi}{2}+i\,sin\frac{\varphi+2k\pi}{2}\right),\quad k=0,1$$

$$\left(\sqrt{\omega}\right)_0=\sqrt{2}\left(cos\frac{3\pi}{4}+i\,sin\frac{3\pi}{4}\right)=-1+i,$$

$$\left(\sqrt{\omega}\right)_1=\sqrt{2}\left(cos\frac{7\pi}{4}+i\,sin\frac{7\pi}{4}\right)=1-i.$$

$$z_1=\frac{(5-i)-(-1+i)}{2(2+i)}=\frac{(5-i)+(1-i)}{2(2+i)}=\cdots=1-i,$$

$$z_2=\frac{(5-i)+(-1+i)}{2(2+i)}=\frac{(5-i)-(1-i)}{2(2+i)}=\cdots=\frac{4}{5}-\frac{2}{5}i.$$

2. Dana je funkcija $g(x) = arc\,cos\left(sin\,3^{\sqrt[3]{-x}}\right)$. Izračunati $g'(-1)$.

Rješenje:

$$g(x) = arc\,cos\left(sin\,3^{\sqrt[3]{-x}}\right).$$

$$g'(x) = -\frac{1}{\sqrt{1-\left(sin\,3^{\sqrt[3]{-x}}\right)^2}} \cdot cos\left(3^{\sqrt[3]{-x}}\right) \cdot \left(3^{\sqrt[3]{-x}}\right)'$$

$$g'(x) = -\frac{1}{cos\left(3^{\sqrt[3]{-x}}\right)} \cdot cos\left(3^{\sqrt[3]{-x}}\right) \cdot 3^{\sqrt[3]{-x}} \cdot (ln\,3) \cdot \frac{1}{3}(-x)^{-\frac{2}{3}} \cdot (-1)$$

$$g'(x) = 3^{\sqrt[3]{-x}} \cdot \frac{1}{3}\frac{1}{\sqrt[3]{(-x)^2}} ln\,3 = \frac{3^{\sqrt[3]{-x}}}{3 \cdot \sqrt[3]{(x^2)}} ln\,3 .$$

Za $x = -1$ slijedi:

$$g'(-1) = \frac{3^{\sqrt[3]{1}}}{3 \cdot \sqrt[3]{(-1)^2}} ln\,3 = \frac{3}{3 \cdot 1} \cdot ln\,3 = ln\,3.$$

3. Analizirati funkciju $f(x) = \frac{x^3 + 2x^2 + 7x - 3}{2x^2}$ i nacrtaj njen graf.

Rješenje:

a) Domena: $D = R \backslash \{0\}$

b) Parnost, neparnost: Nije parna, nije neparna.

c) Nul točke: $y = 0 \Rightarrow x^3 + 2x^2 + 7x - 3 = 0 \Rightarrow x \in (0,1)$
Presjeci s osi y: $x = 0,\ 0 \notin D$

d) Ekstremi:

$$f'(x) = \frac{(3x^2 + 4x + 7)2x^2 - 4x(x^3 + 2x^2 + 7x - 3)}{(2x^2)^2}$$

$$f'(x) = \frac{x^3 - 7x + 6}{2x^3} = \frac{x^3 - x - 6x + 6}{2x^3} = \frac{x(x^2 - 1) - 6(x - 1)}{2x^3}$$

$$f'(x) = \frac{x(x-1)(x+1) - 6(x-1)}{2x^3} = \frac{(x-1)(x^2 + x - 6)}{2x^3} = 0$$

$$\Rightarrow \quad \begin{array}{l} x - 1 = 0 \\ x^2 + x - 6 = 0 \\ \hline x_1 = 1 \end{array}$$

$$x_{2,3}=\frac{-1\pm\sqrt{1+24}}{2}=\frac{-1\pm 5}{2}\Rightarrow \begin{array}{l} x_2=-3 \\ x_3=2 \end{array}$$

x	$-\infty$	-3		0		1		2		$-\infty$
y	↗	M	↘	P	↗	M	↘	m	↗	
y'	+		-		+		-		+	

P - prekid funkcije
m - minimum finkcije
M - maksimum funkcije

$$x=1,\quad f(1)=\frac{1+2+7-3}{2}=\frac{7}{2}\Rightarrow M_1=\left(1,\frac{7}{2}\right)$$

$$x=2,\quad f(2)=\frac{27}{8}\Rightarrow m=\left(2,\frac{27}{8}\right)$$

$$x=-3,\quad f(-3)=-\frac{33}{18}\Rightarrow M_2=\left(-3,\frac{33}{18}\right)$$

e) Infleksija:

$$f''(x)=\frac{\left[\left(x^2+x-6\right)+(x-1)(2x+1)\right]2x^3-6x^2(x-1)\left(x^2+x-6\right)}{4x^6}=$$

$$=\ldots=\frac{7x-9}{x^4}=0\Rightarrow\quad x=\frac{9}{7},\ y=f\left(\frac{9}{7}\right)\Rightarrow\quad I\left(\frac{9}{7},\frac{1307}{378}\right)$$

f) Asimptote:

$$\lim_{x\to\pm 0}\frac{x^3+2x^2+7x-3}{2x^2}=\lim_{x\to\pm 0}\frac{-3}{0}=-\infty$$

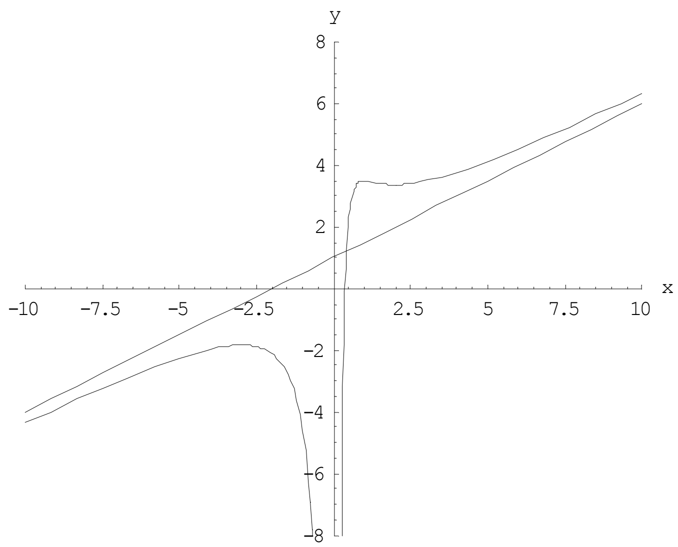
$$\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\frac{x^3+2x^2+7x-3\ \ /:x^3}{2x^3\ \ /:x^3}=\lim_{x\to\infty}\frac{1+\frac{2}{x}+\frac{7}{x^2}-\frac{3}{x^3}}{2}=\frac{1}{2}=k$$

$$\lim_{x\to\infty}\left[f(x)-\frac{1}{2}x\right]=\lim_{x\to\infty}\left[\frac{x^3+2x^2+7x-3}{2x^2}-\frac{1}{2}x\right]=\lim_{x\to\infty}\frac{2x^2+7x-3\ \ /:x^2}{2x^2\ \ /:x^2}=1=l$$

Vertikalna asimptota: $x=0$

Kosa asimptota: $y = \frac{1}{2}x + 1$

g) Graf:



4. Vektor $\vec{c}$ u bazi $(\vec{i}, \vec{j}, \vec{k})$ ima komponente (16, -15, 12). Odrediti koordinate vektora $\vec{d}$ ako je $\vec{d}$ koolinearan s $\vec{c}$, suprotno orijentiran i ako je $|\vec{d}| = 75$.

Rješenje:

$$\vec{d} = \alpha(16,-15,12) = 16\alpha\vec{i} - 15\alpha\vec{j} + 12\alpha\vec{k}$$
$$|\vec{d}| = \sqrt{\alpha^2(16^2 + 15^2 + 12^2)} = \ldots = 25|\alpha|$$
$$|\vec{d}| = 25|\alpha| = 75 \Rightarrow |\alpha| = 3 \Rightarrow \alpha = 3 \vee \alpha = -3$$
$$\vec{d} = -3(16,-15,12) = (-48,45,36)$$

5. Izračunati približno $cos(299°)$ pomoću diferencijala.

Rješenje:

$$f(C + \Delta x) \approx f(C) + f'(C) \cdot \Delta x$$
$$f(C + \Delta x) - f(C) \approx f'(C)dx$$
$$f(x) = cos\,x, \quad C = 300°$$
$$\Delta x = dx = (-1°)\frac{\pi}{180°}$$
$$cos(299°) \approx cos(300°) + (-sin(300°)) \cdot (-1°) = \frac{1}{2} + \frac{\sqrt{3}}{2} \cdot \left(-\frac{\pi}{180}\right) = 0.484885$$
$$cos(299) \approx 0.484885$$

$cos(299°)=0.4848096$

6. Odrediti projekciju točke $T(1,2,8)$ na pravac $p\ldots\begin{cases} x+2y-1=0 \\ y+z=0 \end{cases}$.

Rješenje:

a) Ravnina *Π* točkom *T* okomita na pravac *p*.
b) Probodište *Q* pravca *p* i ravnine *Π*.

Jednadžba pravca:

$$p\ldots\begin{cases} x+2y-1=0 \\ y+z=0 \end{cases} \qquad \begin{aligned} &2y=-x+1 \Rightarrow y=-\frac{x-1}{2} \\ &y=-z \end{aligned}$$

$$p\ldots\frac{x-1}{2}=\frac{y}{-1}=\frac{z}{1} \qquad \vec{p}=\{2,-1,1\},\ P(1,0,0).$$

Ravnina *Π* točkom *T*:

$$\begin{aligned} &2(x-1)-1(y-2)+1(z-8)=0 \\ &2x-2-y+2+z-8=0 \\ &2x-y+z-8=0 \ \ldots \ \Pi. \end{aligned}$$

Probodište:

$$\left.\begin{aligned} x&=2t+1 \\ y&=-t \\ z&=t \end{aligned}\right\} \quad \begin{aligned} &2(2t+1)-(-t)+t-8=0 \\ &4t+2+t+t-8=0 \\ &6t=6 \Rightarrow t=1 \end{aligned} \quad \Rightarrow \quad Q(3,-1,1)$$