Ispitna zadaća 3:

1. Prikazati grafički skup točaka ravnine koji je zadan jednadžbom $x \cdot |x| + y \cdot |y| = 1$.

2. Izračunati $\lim\limits_{x \to 1} x^{\frac{1}{x-1}}$.

3. Ispitati funkciju $f(x) = x - 6ln\left(1 - \frac{1}{x}\right)$ i nacrtati njen graf.

4. Izračunati prve dvije derivacije y', y'' funkcije $y = y(x)$ zadane jednadžbama
$\begin{matrix} x(t) = t^2 + 1 \\ y(t) = t \cdot ln^2 t \end{matrix}$, za vrijednost parametra $t = e^2$.

5. Neka su $A(1,-2,2), B(5,-6,2), C(1,3,-1)$ vrhovi trokuta. Izračunati duljinu visine spuštene iz vrha B na stranicu AC.

6. Naći jednadžbu ravnine s obzirom na koju su pravci
$p \ldots \frac{x-3}{1} = \frac{y+1}{2} = \frac{z-2}{1}$, $\quad q \ldots \frac{x+1}{1} = \frac{y-7}{2} = \frac{z-2}{1}$
međusobno zrcalno simetrični.

Rješenja:

1. Prikazati grafički skup točaka ravnine koji je zadan jednadžbom $x \cdot |x| + y \cdot |y| = 1$.

Rješenje:

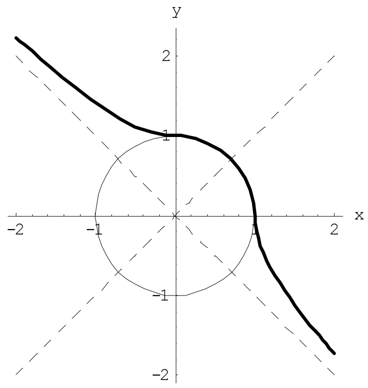
a) $\left.\begin{matrix} x > 0 \\ y > 0 \end{matrix}\right\} \Rightarrow \quad x^2 + y^2 = 1$

b) $\left.\begin{matrix} x > 0 \\ y > 0 \end{matrix}\right\} \Rightarrow \quad x^2 - y^2 = 1$

c) $\left.\begin{matrix} x < 0 \\ y > 0 \end{matrix}\right\} \Rightarrow \quad -x^2 + y^2 = 1$

d) $\left.\begin{matrix} x < 0 \\ y < 0 \end{matrix}\right\} \Rightarrow \quad -x^2 - y^2 = 1 \Rightarrow \quad x^2 + y^2 = -1$

Slika:



2. Izračunati $\lim\limits_{x\to 1} x^{\frac{1}{x-1}}$.

Rješenje:

$\lim\limits_{x\to 1} x^{\frac{1}{x-1}} = 1^{\infty}$, neodređeni oblik

$y = x^{\frac{1}{x-1}} \quad / ln$

$$ln\, y = \frac{1}{x-1} ln\, x \Rightarrow y = e^{\frac{1}{x-1} ln\, x} \quad \Rightarrow \quad lim\, y = e^{\lim\limits_{x\to 1}\left(\frac{1}{x-1} ln\, x\right)}$$

$$\lim_{x\to 1} \frac{ln\, x}{x-1} \quad \left(\frac{0}{0}\right) = \lim_{x\to 1} \frac{\frac{1}{x}}{1} = 1 \quad \Rightarrow \quad \lim_{x\to 1} y = e^{1} = e$$

3. Ispitati funkciju $f(x) = x - 6ln\left(1 - \frac{1}{x}\right)$ i nacrtati njen graf.

Rješenje:

a) Domena:

$$\left(1 - \frac{1}{x}\right) > 0, \quad x \neq 0$$

$$\frac{x-1}{x} > 0 \quad \Rightarrow \quad \left\{ \begin{array}{l} x-1>0 \\ \underline{x>0} \\ x>1 \\ \underline{x>0} \\ x>1 \end{array} \quad \text{ili} \quad \begin{array}{l} x-1<0 \\ \underline{x<0} \\ x<1 \\ \underline{x<0} \\ x<0 \end{array} \right\} \quad \Rightarrow$$

$D = \langle -\infty, 0\rangle \cup \langle 1, +\infty\rangle$

b) Parnost i neparnost: Nije parna, nije neparna (domena nije simetrična prema ishodištu).

c) Nul točke: $y = 0 \Rightarrow \frac{x}{6} = ln\left(1 - \frac{1}{x}\right) \Rightarrow$ nema točaka presjeka

Presjeci s osi y: $x = 0,\ \ 0 \notin D$

d) Ekstremi:

$$f'(x) = 1 - \frac{6}{1 - \frac{1}{x}} \cdot \frac{1}{x^2} = 1 - \frac{6}{(x-1)x}$$

$$f'(x) = \frac{x(x-1) - 6}{x(x-1)} = \frac{x^2 - x - 6}{x(x-1)} = 0 \quad \Rightarrow \quad x^2 - x - 6 = 0$$

$$x_{1,2} = \frac{1 \pm \sqrt{1+24}}{2} = \frac{1 \pm 5}{2} \quad \Rightarrow \quad x_1 = -2,\ x_2 = 3$$

x	$-\infty$	-2		0	1		3		$-\infty$
y	↗	M	↘	P		↘	m	↗	
y'	+		-			-		+	

P - prekid funkcije
m - minimum finkcije
M - maksimum funkcije

$$f(x_1) = f(-2) = -2 - 6\,ln\left(\frac{3}{2}\right) = -4.43279 \quad \Rightarrow \quad M(-2, -4.43279)$$

$$f(x_2) = f(3) = 3 - 6\,ln\left(\frac{2}{3}\right) = 5.43279 \quad \Rightarrow \quad m(3, 5.43279)$$

e) Infleksija:

$$f''(x) = \frac{(2x-1)x(x-1) - (2x-1)(x^2 - x - 6)}{x^2(x-1)^2} = \frac{(2x-1)(x^2 - x) - (2x-1)(x^2 - x - 6)}{x^2(x-1)^2}$$

$$f''(x) = \frac{(2x-1)(x^2 - x - x^2 + x + 6)}{x^2(x-1)^2} = \frac{6(2x-1)}{x^2(x-1)^2} = 0$$

$$2x - 1 = 0 \quad \Rightarrow \quad x = \frac{1}{2},\ \ \frac{1}{2} \notin D \ \Rightarrow$$ nema točaka infleksije

f) Asimptote:

$$\lim_{x \to 0^-}\left(x - 6\,ln\left(1 - \frac{1}{x}\right)\right) = 0 - \infty = -\infty$$

$$\lim_{x\to 1^+}\left(x-6\,ln\left(1-\frac{1}{x}\right)\right)=\left(1-6\,ln\frac{0}{1}\right)=1-(-\infty)=+\infty$$

Vertikalne asimptote: $x=0,\quad x=1$

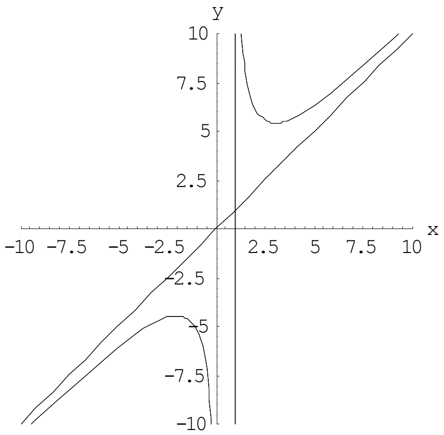
$$k=\lim_{x\to\infty}=\left(\frac{f(x)}{x}\right)=\lim_{x\to\infty}\left(1-\frac{6\,ln\left(\frac{x-1}{x}\right)}{x}\right)=\lim_{x\to\infty}\left(1-\frac{6\,ln\left(1-\frac{1}{x}\right)}{x}\right)=$$

$$=\lim_{x\to\infty}\left(1-\frac{1}{x}\cdot 6\,ln\left(1-\frac{1}{x}\right)\right)=1=k$$

$$l=\lim_{x\to\infty}(f(x)-kx)=\lim_{x\to\infty}\left(x-6\,ln\left(1-\frac{1}{x}\right)-x\right)=0=l$$

Kosa asimptota: $y=x$

g) Graf:



4. Izračunati prve dvije derivacije y', y'' funkcije $y=y(x)$ zadane jednadžbama
$\begin{matrix} x(t)=t^2+1 \\ y(t)=t\cdot ln^2 t \end{matrix}$, za vrijednost parametra $t=e^2$.

Rješenje:

$$y'(x)=\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\dot{y}}{\dot{x}}=\frac{ln^2 t+t\cdot 2\,ln\,t\cdot\frac{1}{t}}{2t}=\frac{ln^2 t+2\,ln\,t}{2t}$$

$$y''(x)=\frac{dy'}{dx}=\frac{\frac{dy'}{dt}}{\frac{dx}{dt}}=\frac{\ddot{y}\dot{x}-\ddot{x}\dot{y}}{(\dot{x})^3}=\frac{\left(2\,ln\,t\cdot\frac{1}{t}+\frac{2}{t}\right)2t-2\left(ln^2 t+2\,ln\,t\right)}{4t^2}\cdot\frac{1}{2t}$$

$$y''(x) = \frac{2\,ln\,t + 2 - ln^2\,t - 2\,ln\,t}{4t^3} = \frac{2 - ln^2\,t}{4t^3}.$$

$$t = e^2 \quad \Rightarrow \quad y'(x) = \frac{4}{e^2}, \quad y''(x) = -\frac{1}{2e^6}.$$

5. Neka su $A(1,-2,2)$, $B(5,-6,2)$, $C(1,3,-1)$ vrhovi trokuta. Izračunati duljinu visine spuštene iz vrha B na stranicu AC.

Rješenje:

1. način:

$$\left|\overrightarrow{AC}\right| \cdot h = \left|\overrightarrow{AC} \times \overrightarrow{AB}\right|, \quad h = ?$$

$$\left|\overrightarrow{AC}\right| = \left|\{0,4,-3\}\right| = \sqrt{16+9} = 5$$

$$\left|\overrightarrow{AB}\right| = \left|\{4,-5,0\}\right| = \sqrt{16+25} = \sqrt{41}$$

$$\overrightarrow{AC} \times \overrightarrow{AB} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 4 & -3 \\ 4 & -5 & 0 \end{vmatrix} = -15\vec{i} + 12\vec{j} + 16\vec{k}$$

$$\left|\overrightarrow{AC} \times \overrightarrow{AB}\right| = \sqrt{15^2 + 12^2 + 16^2} = 25 \quad \Rightarrow \quad h = \frac{25}{5}$$

2. način:

$$\left.\begin{aligned} \overrightarrow{AB} \cdot \overrightarrow{AC} &= \left|\overrightarrow{AB}\right| \cdot \left|\overrightarrow{AC}\right| cos\,\alpha = \sqrt{41} \cdot 5\,cos\,\alpha \\ \overrightarrow{AB} \cdot \overrightarrow{AC} &= \{4,-5,0\} \cdot \{0,4,-3\} = -20 \end{aligned}\right\} \Rightarrow cos\,\alpha = -\frac{4}{\sqrt{41}}$$

$$\left|\overrightarrow{AD}\right| = \left|\overrightarrow{AB} \cdot \frac{\overrightarrow{AC}}{\left|\overrightarrow{AC}\right|}\right| = \left|\{4,-5,0\} \cdot \left\{0, \frac{4}{5}, -\frac{3}{5}\right\}\right| = \left|-4\right| = 4 \quad \dots$$

duljina projekcije vektora $\overrightarrow{AB}$ na vektor $\overrightarrow{AC}$

$$h^2 = \left|\overrightarrow{AB}\right|^2 - \left|\overrightarrow{AD}\right|^2 = 41 - 16 = 25 \Rightarrow \quad h = 5$$

6. Naći jednadžbu ravnine s obzirom na koju su pravci

$$p \ldots \frac{x-3}{1} = \frac{y+1}{2} = \frac{z-2}{1}, \qquad q \ldots \frac{x+1}{1} = \frac{y-7}{2} = \frac{z-2}{1}$$

međusobno zrcalno simetrični.

Rješenje:

$$\left.\begin{aligned}\vec{p} &= \{1,2,1\}\\ \vec{q} &= \{1,2,1\}\end{aligned}\right\} \Rightarrow \vec{p} \parallel \vec{q}$$

a) Ravnina *Π* okomito na $\vec{p}\, i\, \vec{q}$, točkom *P*:

$\vec{n} = \vec{p} = \vec{q} = \{1,2,1\} \qquad P(3,-1,2), \quad P \in p$

$$\begin{aligned}\Pi \ldots\ & 1(x-3)+2(y+1)+1(z-2)=0\\ & x-3+2y+2+z-2=0\\ \Pi \ldots\ & x+2y+z-3=0\end{aligned}$$

b) Probodište *R* pravca q i ravnine Π :

$$q \ldots \begin{cases} x = t-1 \\ y = 2t+7 \\ z = t+2 \end{cases} \qquad \begin{aligned} & t-1+2(2t+7)+(t+2)-3=0 \\ & t+4t+t-1+14+2-3=0 \\ & 6t+12=0 \Rightarrow t=-2 \Rightarrow R(-3,3,0) \end{aligned}$$

$\overrightarrow{PR} = \{-3-3, 3+1, 0-2\} = \{-6,4,-2\} = \{3,-2,1\}$

Polovište dužine $\overline{PR}$: $S(0,1,1)$

c) Ravnina *Σ* točkom *S*, vektora normale $\overrightarrow{PR}$:

$3(x-0)-2(y-1)+1(z-1)=0$

$3x-2y+z+1=0 \ldots \Sigma$