Ispitna zadaća 5:

1. Odrediti realnu nulu polinoma $P_3(x) = x^3 + ax + b$ i koeficijente $a,\ b \in R$, ako je poznato da je $P_3(1-2i) = 0$.

2. Odrediti domenu i derivaciju funkcije $y = \frac{1}{3} ln|x+1| - \frac{1}{6} ln(x^2 - x + 1) + \frac{1}{\sqrt{3}} arctg \frac{2x-1}{\sqrt{3}}$.

3. Odrediti domenu, nul točke, ekstreme, intervale monotonosti i asimptote funkcije $y = \frac{x^2 - 6}{x(x^2 - 4)}$. Skicirati njen graf.

4. Iz točke na osi ordinata povući tangente na krivulju $y = -x^2 + 4$ tako da s osi apscisa zatvaraju trokut minimalne površine. Izračunati površinu tog trokuta.

5. Izračunati površinu paralelograma čije su dijagonale $\vec{e} = \vec{m} + 2\vec{n}$ i $\vec{f} = 3\vec{m} + \vec{n}$, gdje je $|\vec{n}| = 1,\ |\vec{m}| = 2$ a $\vec{m}$ i $\vec{n}$ zatvaraju kut od 60°.

6. Odrediti kut između pravca $p \ldots \begin{cases} x + y + z - 2 = 0 \\ 2x + y - z - 1 = 0 \end{cases}$ i ravnine koja prolazi točkama $A(2,3,-1),\ B(1,1,0),\ C(0,-2,1)$.

Rješenja:

1. Odrediti realnu nulu polinoma $P_3(x) = x^3 + ax + b$ i koeficijente $a,\ b \in R$, ako je poznato da je $P_3(1-2i) = 0$.

Rješenje:

$$P_3(x) = (x - x_1)(x - x_2)(x - x_3) = x^3 + ax + b \quad \Rightarrow \quad \frac{x^3 + ax + b}{(x - x_1)(x - x_2)} = (x - x_3)$$

$$x_1 = 1 - 2i$$
$$x_2 = 1 + 2i$$

$$x - x_1 = x - 1 + 2i$$
$$x - x_2 = x - 1 - 2i$$

$$\begin{aligned}(x - x_1)(x - x_2) &= (x - (1 - 2i))(x - (x + 2i)) = \\ &= x^2 - (1 + 2i)x - (1 - 2i)x + (1 - 2i)(1 + 2i) = \\ &= x^2 - 2x + 5\end{aligned}$$

1. način:

$$x^3 + ax + b = \left(x^2 - 2x + 5\right)\left(x - x_3\right) =$$
$$= x^3 - \left(x_3 + 2\right)x^2 + \left(2x_3 + 5\right)x - 5x_3$$

$$x_3 + 2 = 0 \quad \Rightarrow \quad x_3 = -2$$
$$\Rightarrow \quad 2x_3 + 5 = a \quad \Rightarrow \quad a = 1$$
$$-5x_3 = b \quad \Rightarrow \quad b = 10$$

$$\Rightarrow \quad P_3(x) = \left(x - (1 - 2i)\right)\left(x - (1 + 2i)\right)\left(x + 2\right) \square$$

2. način:

$$\left(x^3 + ax + b\right) : \left(x^2 - 2x + 5\right) = x + 2 = x - x_3$$
$$\underline{x^3 - 2x^2 + 5x}$$

$$2x^2 + ax - 5x + b$$
$$2x^2 + x(a - 5) + b$$
$$\underline{2x^2 - 4x + 10}$$

$$ax - 5x + 4x + b - 10$$
$$ax - x + b - 10 = 0$$
$$x(a - 1) = 10 - b \Rightarrow a = 1,\ b = 10$$

$$P_3(x) = x^3 + x + 10$$
$$P_3(x) = \left(x - (1 - 2i)\right)\left(x - (1 + 2i)\right)\left(x + 2\right) \square$$

2. Odrediti domenu i derivaciju funkcije $y = \frac{1}{3} ln\left|x + 1\right| - \frac{1}{6} ln\left(x^2 - x + 1\right) + \frac{1}{\sqrt{3}} arctg \frac{2x - 1}{\sqrt{3}}$.

Rješenje:

Domena:

a)

$$y = ln\left(\left|x + 1\right|\right)$$

$$x + 1 > 0$$
$$\underline{x > -1}$$

b)

$$y = ln\left(x^2 - x + 1\right)$$

$$x^2 - x + 1 > 0$$
$$\underline{x_{1,2} = \frac{1 \pm \sqrt{1 - 4}}{2} \notin R}$$

c)

$$y = arctg\left(\frac{2x - 1}{\sqrt{3}}\right)$$

$$\underline{x \in R}$$

$-(x+1)>0$

$-x-1>0$ $\qquad\qquad$ $\underline{x \in R}$

$\underline{x<-1}$

$\underline{x \in R \setminus \{-1\}}$

a) & b) & c) $\Rightarrow$ $D = R \setminus \{-1\}$

Derivacija:

$$x>-1 \quad \Rightarrow \quad y' = \frac{1}{3(x+1)} - \frac{2x-1}{6(x^2-x+1)} + \frac{1}{\sqrt{3}} \cdot \frac{1}{1+\frac{(2x-1)^2}{3}} \cdot \frac{2}{\sqrt{3}} =$$

$$= \frac{(2x^2-2x+2)-(2x^2+x-1)}{6(x^3+1)} + \frac{2}{3+4x^2-4x+1} =$$

$$= \frac{(-x+1)+(x+1)}{2(x^3+1)} = \frac{1}{x^3+1}$$

$$x<-1 \quad \Rightarrow \quad y' = \frac{1}{3(-x-1)}(-1) - \frac{2x-1}{6(x^2-x+1)} + \frac{1}{\sqrt{3}} \cdot \frac{1}{1+\frac{(2x-1)^2}{3}} \cdot \frac{2}{\sqrt{3}} =$$

$$= \ldots = \frac{1}{x^3+1} \square$$

3. Odrediti domenu, nul točke, ekstreme, intervale monotonosti i asimptote funkcije $y = \frac{x^2-6}{x(x^2-4)}$. Skicirati njen graf.

Rješenje:

a) Domena:

$x(x^2-4) \neq 0 \quad \Rightarrow \quad x \neq 0 \quad \& \quad x \neq \pm 2$

$D = R \setminus \{-2, 0, 2\}$

b) Parnost, neparnost:

$$f(-x) = \frac{x^2-6}{-x(x^2-4)}, \quad -f(x) = -\frac{x^2-6}{x(x^2-4)} = f(-x)$$

Funkcija je neparna.

c) Nul točke: $\qquad y=0 \Rightarrow x^2-6=0, \; x = \pm\sqrt{6}$

Presjeci s osi y: $\qquad x=0 \Rightarrow 0 \notin D$

d) Ekstremi:

$$y' = \frac{2x \cdot x\left(x^2 - 4\right) - \left(x^2 - 6\right)\left[x^2 - 4 + x(2x)\right]}{x^2\left(x^2 - 4\right)}^2$$

$$y' = \frac{2x^2\left(x^2 - 4\right) - \left(x^2 - 6\right)\left(3x^2 - 4\right)}{x^2\left(x^2 - 4\right)^2}$$

$$y' = \frac{-x^4 + 14x^2 - 24}{x^2\left(x^2 - 4\right)^2} = 0$$

$$x^4 - 14x + 24 = 0$$

$$\left(x^2\right)_{1,2} = \frac{14 \pm \sqrt{14^2 - 4 \cdot 24}}{2}$$

$$\left(x^2\right)_{1,2} = \frac{14 \pm \sqrt{196 - 96}}{2} = \frac{14 \pm 10}{2}$$

$$\left(x^2\right)_1 = 2 \Rightarrow x_{1,2} = \pm\sqrt{2}$$

$$\left(x^2\right)_2 = 12 \Rightarrow x_{3,4} = \pm 2\sqrt{3}$$

x	$-\infty$		$-2\sqrt{3}$		-2		$-\sqrt{2}$		0		$\sqrt{2}$		2		$2\sqrt{3}$		∞
y		↘	m	↗	P	↗	M	↘	P	↘	m	↗	P	↗	M	↘	
y'		-		+		+		-		-		+		+		-	

P - prekid funkcije
m - minimum finkcije
M - maksimum funkcije

Neparnost funkcije nam dozvoljava da gledamo samo pozitivni dio domene:

$$x_1 = \sqrt{2} \quad \Rightarrow \quad f\left(x_1\right) = \sqrt{2} \quad \Rightarrow \quad m_1\left(\sqrt{2}, \sqrt{2}\right), \qquad M_1\left(-\sqrt{2}, -\sqrt{2}\right)$$

$$x_3 = 2\sqrt{3} \Rightarrow f\left(x_2\right) = \frac{\sqrt{3}}{8} \quad \Rightarrow \quad M_2\left(2\sqrt{3}, \frac{\sqrt{3}}{8}\right), \quad m_2\left(-2\sqrt{3}, -\frac{\sqrt{3}}{8}\right)$$

e) Asimptote:

$$\lim_{x \to 0^+} \frac{x^2 - 6}{x\left(x^2 - 4\right)} = \infty$$

$$\lim_{x \to 2^+} \frac{x^2 - 6}{x\left(x^2 - 4\right)} = -\infty, \quad \lim_{x \to 2^-} \frac{x^2 - 6}{x\left(x^2 - 4\right)} = \infty$$

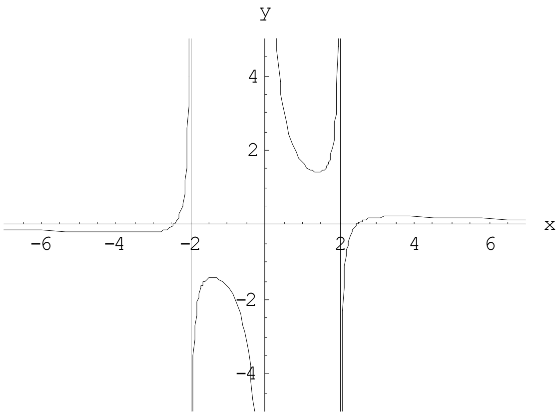
Vertikalne asimptote: $x = 0, \quad x = \pm 2$

$$\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\frac{x^2-6}{\left(x^2-4\right)\cdot x^2}=0=k$$

$$\lim_{x\to\pm\infty}f(x)=\lim_{x\to\pm\infty}\frac{x^2-6}{x^3-4x}=\lim_{x\to\pm\infty}\frac{\frac{1}{x}-\frac{6}{x^3}}{1-\frac{4}{x^2}}=\frac{0-0}{1-0}=0=l$$

Horizontalna asimptota: $y=0$

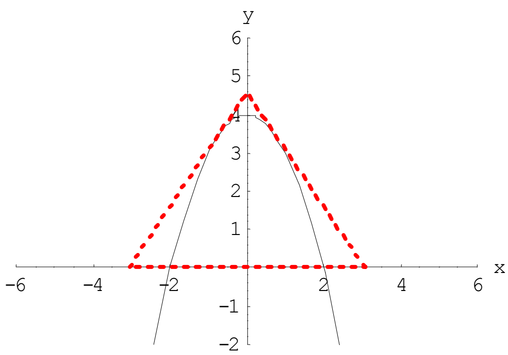
f) Graf:



□

4. Iz točke na osi ordinata povući tangente na krivulju $y=-x^2+4$ tako da s osi apscisa zatvaraju trokut minimalne površine. Izračunati površinu tog trokuta.

Rješenje:



$$y=-x^2+4$$
$$y'=-2x$$

Jednadžba tangente parabole $y=-x^2+4$ u točki (x_0,y_0):

$$y-y_0=y'(x_0)(x-x_0)$$
$$y-\left(-x_0^2+4\right)=-2x_0(x-x_0)$$
$$y+x_0^2-4=-2x_0(x-x_0)$$
$$\underline{y=-2xx_0+x_0^2+4}$$

Presjeci tangente s koordinatnim osima:

$$y=0 \Rightarrow m \dots \; -2xx_0+x_0^2+4=0$$

$$x=\frac{x_0^2+4}{2x_0}$$

$$x=0 \Rightarrow n \dots \; y=x_0^2+4$$

$$P(x_0)=|m\cdot n|=\left|\frac{x_0^2+4}{2x_0}\cdot\left(x_0^2+4\right)\right|=\left|\frac{\left(x_0^2+4\right)^2}{2x_0}\right|$$

$$P'(x_0)=\frac{2\left(x_0^2+4\right)\cdot 2x_0\cdot 2x_0-2\left(x_0^2+4\right)^2}{4x_0^2}=\frac{\left(x_0^2+4\right)\left(3x_0^2-4\right)}{2x_0^2}=0$$

$$x_0^2+4=0 \notin R$$

$$3x_0^2-4=0 \Rightarrow x_0^2=\frac{4}{3} \;\Rightarrow\; x_0=\pm\frac{2\sqrt{3}}{3}$$

$$P=\frac{\left(\frac{4}{3}+4\right)^2}{2\cdot\frac{2\sqrt{3}}{3}}=\frac{32}{3} \;\square$$

5. Izračunati površinu paralelograma čije su dijagonale $\vec{e}=\vec{m}+2\vec{n}$ i $\vec{f}=3\vec{m}+\vec{n},$ gdje je $\left|\vec{n}\right|=1, \left|\vec{m}\right|=2$ a $\vec{m}$ i $\vec{n}$ zatvaraju kut od 60°.

Rješenje:

$\vec{a}$ i $\vec{b}$ su stranice paralelograma

$$\left.\begin{aligned}\vec{a}+\vec{b}&=\vec{e}\\ \vec{a}-\vec{b}&=\vec{f}\end{aligned}\right\}\Rightarrow \begin{aligned}\vec{a}&=\frac{\vec{e}+\vec{f}}{2}\\ \vec{b}&=\frac{\vec{e}-\vec{f}}{2}\end{aligned}$$

$$P=\left|\vec{a}\times\vec{b}\right|=\left|\frac{1}{2}\left(\vec{e}+\vec{f}\right)\times\frac{1}{2}\left(\vec{e}-\vec{f}\right)\right|=\frac{1}{4}\left|2\left(\vec{f}\times\vec{e}\right)\right|$$

$$P=\frac{1}{2}\left|\left(\vec{m}+2\vec{n}\right)\times\left(3\vec{m}+\vec{n}\right)\right|=\frac{1}{2}\left|3\vec{m}\times\vec{m}+\vec{m}\times\vec{n}+2\vec{n}\times3\vec{m}+2\vec{n}\times\vec{n}\right|$$

$$P=\frac{5}{2}\left|\vec{n}\times\vec{m}\right|=\frac{5}{2}\left|\vec{n}\right|\cdot\left|\vec{m}\right|\cdot sin\left[\angle\left(\vec{m},\vec{n}\right)\right]=\frac{5}{2}\cdot1\cdot2\cdot\frac{\sqrt{3}}{2}=\frac{5\sqrt{3}}{2} \;\square$$

6. Odrediti kut između pravca $p \ldots \begin{cases} x+y+z-2=0 \\ 2x+y-z-1=0 \end{cases}$ i ravnine koja prolazi točkama $A(2,3,-1), B(1,1,0), C(0,-2,1)$.

Rješenje:

Ravnina trima točkama:

$$\overrightarrow{AB}=\{-1,-2,1\}$$
$$\overrightarrow{AC}=\{-2,-5,2\}$$
$$\overrightarrow{AT}=\{x-2, y-3, z+1\}$$

$$\begin{vmatrix} x-2 & y-3 & z+1 \\ -1 & -2 & 1 \\ -2 & -5 & 2 \end{vmatrix}=0$$

$$(x-2)(-4+5)-(y-3)(-2+2)+(z+1)(5-4)=0$$

$$\Pi \ldots \; x+z-1=0 \qquad \vec{n}=\{1,0,1\}$$

Pravac u kanonskom (i parametarskom) obliku:

$$\overrightarrow{n_1}=\{1,1,1\}, \quad \overrightarrow{n_2}=\{2,1,-1\}, \quad \vec{p}=\overrightarrow{n_1}\times\overrightarrow{n_2}$$

$$\vec{p}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{vmatrix}=-2\vec{i}+3\vec{j}-\vec{k}, \qquad \vec{p}=\{-2,3,-1\},$$

$$|\vec{p}|=\sqrt{4+9+1}=\sqrt{14}$$

$$x=0 \Rightarrow \left.\begin{matrix} y+z-2=0 \\ y-z-1=0 \end{matrix}\right\} \Rightarrow \begin{cases} y=\frac{3}{2} \\ z=\frac{1}{2} \end{cases} \Rightarrow P\left(0,\frac{3}{2},\frac{1}{2}\right)$$

$$p \ldots \frac{x}{-2}=\frac{y-\frac{3}{2}}{3}=\frac{z-\frac{1}{2}}{-1}, \qquad p \ldots \begin{cases} x=-2t \\ y=3t+\frac{3}{2} \\ z=-t+\frac{1}{2} \end{cases}$$

1. način

Kut β između normale ravnine Π i pravca p:

$$\cos\beta=\frac{\vec{n}\,\vec{p}}{|\vec{n}||\vec{p}|}=-\frac{3}{2\sqrt{7}}$$

Traženi kut α jest:

$$\alpha=\arccos\left(-\frac{3}{2\sqrt{7}}\right)-\frac{\pi}{2}=34^\circ\ 32'\,15''$$

2. način

Probodište pravca p i ravnine Π :

$$p\ldots\begin{cases} x=-2t \\ y=3t+\dfrac{3}{2}, \\ z=-t+\dfrac{1}{2}\end{cases} \quad \& \quad x+z-1=0\ \ldots\Pi \quad \Rightarrow \quad t=-\frac{1}{6} \quad \Rightarrow \quad Q\left(\frac{1}{3},1,\frac{2}{3}\right)$$

Normala točkom P:

$$P\left(0,\frac{3}{2},\frac{1}{2}\right) \quad \& \quad \vec{n}=\{1,0,1\} \quad \Rightarrow \quad \frac{x}{1}=\frac{y-\frac{3}{2}}{0}=\frac{z-\frac{1}{2}}{1}\ \ldots\ n$$

Probodište normale i ravnine Π :

$$n\ldots\begin{cases} x=t \\ y=\dfrac{3}{2} \\ z=t+\dfrac{1}{2}\end{cases} \quad \& \quad \Pi\ldots\ x+z-1=0 \quad \Rightarrow \quad t=\frac{1}{4} \quad \Rightarrow \quad R\left(\frac{1}{4},\frac{3}{2},\frac{3}{4}\right)$$

Vektor $\overrightarrow{QR}$:

$$\overrightarrow{QR}=\left(\frac{1}{4}-\frac{1}{3}\right)\vec{i}+\left(\frac{3}{2}-1\right)\vec{j}+\left(\frac{3}{4}-\frac{2}{3}\right)\vec{k}$$

$$\overrightarrow{QR}=-\frac{1}{12}\vec{i}+\frac{1}{2}\vec{j}+\frac{1}{12}\vec{k}, \qquad \left|\overrightarrow{QR}\right|=\frac{\sqrt{38}}{12}$$

Kut između vektora $\vec{p}$ & $\overrightarrow{QR}$:

$$\vec{p}\cdot\overrightarrow{QR}=\left|\vec{p}\right|\cdot\left|\overrightarrow{QR}\right|\cdot\cos\alpha=\sqrt{14}\cdot\frac{\sqrt{38}}{12}\cdot\cos\alpha$$

$$\vec{p}\cdot\overrightarrow{QR}=\{-2,3,-1\}\cdot\left\{-\frac{1}{12},\frac{1}{2},\frac{1}{12}\right\}=\frac{2}{12}+\frac{3}{2}-\frac{1}{12}=\frac{19}{12}$$

$$\frac{\sqrt{38\cdot 14}}{12}\cos\alpha=\frac{19}{12}\Rightarrow\cos\alpha=\frac{19}{\sqrt{38\cdot 14}}=\frac{\sqrt{19\cdot 7}}{14}\Rightarrow\alpha=34^\circ 32' 15'' \ \square$$