Ispitna zadaća 7:

1. Bez i s L'Hospitalovim pravilom izračunati $\lim\limits_{x\to 1}\dfrac{x+x^2+x^3+...+x^n-n}{x-1}$.

2. Zadana je elipsa $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. Pokazati da je produkt udaljenosti obaju fokusa do tangente u proizvoljnoj točke $T_0(x_0,y_0)$ konstantan i jednak kvadratu male poluosi elipse.

3. Naći domenu, presjek s koordinatnim osima, ekstreme, asimptote i skicirati graf funkcije $y=e^{\frac{1}{x^2+4x+3}}$.

4. Od triju dasaka širine a treba sastaviti korito najvećeg protoka.

5. Funkciju $f(x)=\dfrac{1}{\sqrt{x}}$ aproksimirati Taylorovim polinomom 3. stupnja u okolini točke $a=1$ i skicirati graf zadane funkcije i dobivenog polinoma.

6. Odabrati m tako da se pravci $\dfrac{x-2}{3}=\dfrac{y-4}{1}=\dfrac{z-2}{1}$ i $\begin{cases} x-z+2=0 \\ y-mz-1=0 \end{cases}$ sijeku i odrediti ravninu koju određuju.

Rješenja:

1. Bez i s L'Hospitalovim pravilom izračunati $\lim\limits_{x\to 1}\dfrac{x+x^2+x^3+...+x^n-n}{x-1}$.

Rješenje:

Bez L´Hospitalovog pravila:

$$\lim_{x\to 1}\frac{x+x^2+x^3+...+x^n-n}{x-1} = \lim_{x\to 1}\frac{(x-1)+(x^2-1)+(x^3-1)+...+(x^n-1)}{x-1} =$$

$$= \lim_{x\to 1}\frac{(x-1)\left(1+(x+1)+(x^2+x+1)+\cdots+(x^{n-1}+x^{n-2}+\cdots+x+1)\right)}{x-1} =$$

$$= \lim_{x\to 1}\left(1+(x+1)+(x^2+x+1)+\cdots+(x^{n-1}+x^{n-2}+\cdots+x+1)\right) =$$

$$= \lim_{x\to 1}\left(n\cdot 1+(n-1)x+(n-2)x^2+\cdots+x^{n-1}\right) = 1+2+...+n = \frac{n}{2}(n+1)$$

Primjenom L'Hospitalovog pravila:

$$\lim_{x\to 1} \frac{1+2x+3x^2+...+nx^{n-1}}{1} = 1+2+...+n = \frac{n}{2}(n-1)□$$

2. Zadana je elipsa $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Pokazati da je produkt udaljenosti obaju fokusa do tangente u proizvoljnoj točke $T_0(x_0, y_0)$ konstantan i jednak kvadratu male poluosi elipse.

Rješenje:

Jednadžba elipse: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Jednadžba tangente u točki $T_0(x_0, y_0)$: $\frac{xx_0}{a^2}+\frac{yy_0}{b^2}=1$

$$x_0b^2x+y_0a^2y=a^2b^2$$

Koeficijenti: $A=x_0b^2$, $B=ab\sqrt{a^2-x_0^2}$, C=-a^2b^2

Ekscentricitet: $e=\sqrt{a^2-b^2}$

Fokusi: $F_1(-e,0)$, $F_2(e,0)$

Udaljenosti izračunati po formuli: $d_i=\frac{Ax_{Fi}+By_{Fi}+C}{\sqrt{A^2+B^2}}$

$$d_1=-\frac{x_0b^2\sqrt{a^2-b^2}-a^2b^2}{\sqrt{x_0^2b^4+a^2b^2(a^2-x_0^2)}}=-\frac{b(x_0\sqrt{a^2-b^2}-a^2)}{\sqrt{x^2{}_0b^2+a^4-a^2x_0{}^2}}$$

$$d_2=\frac{b(x_0\sqrt{a^2-b^2}+a^2)}{\sqrt{x_0^2b^2+a^4-a^2x_0^2}}$$

$$d_1d_2=b^2□$$

3. Naći domenu, presjek s koordinatnim osima, ekstreme, asimptote i skicirati graf funkcije $y=e^{\frac{1}{x^2+4x+3}}$.

Rješenje:

a) Domena:

$x^2+4x+3\neq 0$

$x^2+4x+3=0 \Rightarrow x_1=-1,\ x_2=-3 \Rightarrow D=R\backslash\{-1, -3\}$

b) Parnost, neparnost: Nije parna, nije neparna (domena nije simetrična prema ishodištu).

c) Nul točke: $e^x \neq 0, \forall x \Rightarrow$ nema nul točaka

Presjeci s osi y: $x = 0 \Rightarrow y = e^{\frac{1}{3}} \Rightarrow \left(0, e^{\frac{1}{3}}\right)$

d) Ekstremi:

$$f'(x) = -e^{\frac{1}{x^2+4x+3}} \frac{2x+4}{\left(x^2+4x+3\right)^2} = 0 \Rightarrow -2x = 4 \Rightarrow x = -2$$

x	-∞	-3	-2	-1	+∞
y	↗	P ↗	M ↘	P ↘	
y'	+	+	-	-	

P - prekid funkcije
m - minimum finkcije
M - maksimum funkcije

$$x = -2, \quad f(-2) = \frac{1}{e} \Rightarrow M_{max}\left(-2, \frac{1}{e}\right)$$

e) Asimptote:

$$\lim_{x \to (-1)^+} e^{\frac{1}{x^2+4x+3}} = e^{\lim_{x \to (-1)^+} \frac{1}{x^2+4x+3}} = e^{\lim_{x \to 1^+} \frac{1}{(-x)^2-4x+3}} = e^{+\infty} = +\infty$$

$$\lim_{x \to (-1)^-} e^{\frac{1}{x^2+4x+3}} = 0 = e^{\lim_{x \to 1^-} \frac{1}{(-x)^2-4x+3}} = e^{\lim_{x \to 1^-} \frac{1}{x^2-4x+3}} = e^{-\infty} = \frac{1}{e^{\infty}} = 0$$

$$\lim_{x \to (-3)^+} e^{\frac{1}{x^2+4x+3}} = e^{\lim_{x \to (3)^-} \frac{1}{(x)^2-4x+3}} = e^{-\infty} = 0$$

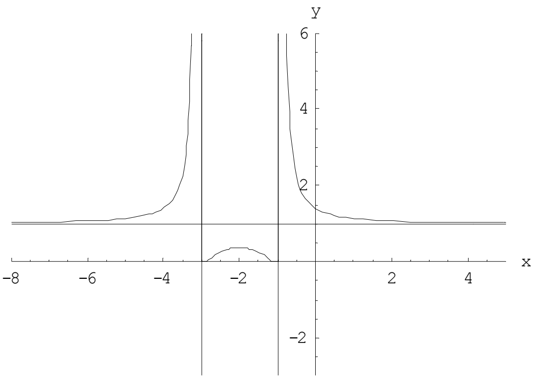
$$\lim_{x \to (-3)^-} e^{\frac{1}{x^2+4x+3}} = e^{\lim_{x \to (3)^+} \frac{1}{(x)^2-4x+3}} = e^{+\infty} = +\infty$$

$$\lim_{x \to \pm\infty} e^{\frac{1}{x^2+4x+3}} = e^0 = 1$$

Vertikalne asimptote: $x = -1, \; x = -3$

Horizontalna asimptota: $y = 1$

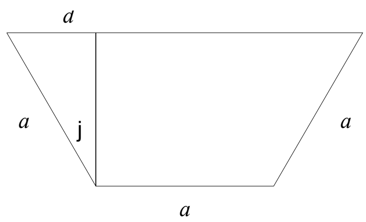
f) Graf:



□

4. Od triju dasaka širine a treba sastaviti korito najvećeg protoka.

Rješenje:



$$c = a + 2d$$
$$v = a\cos\varphi$$
$$d = a\sin\varphi$$

Površina trapeza: $P = \dfrac{(a+c)v}{2}$

$$P = \frac{(a + a + 2a\,sin\,\varphi)a\,cos\,\varphi}{2}$$
$$P(\varphi) = a^2\,cos\,\varphi + a^2\,sin\,\varphi\,cos\,\varphi$$
$$P'(\varphi) = -a^2\,sin\,\varphi + a^2\left(cos^2\,\varphi - sin^2\,\varphi\right)$$
$$P'(\varphi) = 0 \quad \Rightarrow \quad -\,sin\,\varphi + cos^2\,\varphi - sin^2\,\varphi = 0$$
$$-\,sin\,\varphi + \left(1 - sin^2\,\varphi\right) - sin^2\,\varphi = 0$$
$$-\,2\,sin^2\,\varphi - sin\,\varphi + 1 = 0$$
$$(sin\,\varphi)_1 = \frac{1}{2} \Rightarrow \varphi_1 = \frac{\pi}{6},$$
$$(sin\,\varphi)_2 = -1 \Rightarrow \varphi_2 = \frac{3\pi}{2},$$

$$P''(\varphi) = -a^2\,cos\,\varphi - 2a^2\,sin\,2\varphi$$

$$P''(\varphi_1) < 0 \quad \Rightarrow \quad \text{maksimum}$$
$$P''(\varphi_2) > 0 \quad \Rightarrow \quad \text{minimum}$$

Korito najvećeg protoka dobije se ako se daske postave pod kutom od 120^{o}.□

5. Funkciju $f(x)=\dfrac{1}{\sqrt{x}}$ aproksimirati Taylorovim polinomom 3. stupnja u okolini točke $a = 1$ i skicirati graf zadane funkcije i dobivenog polinoma.

Rješenje:

$f(x)=x^{-\frac{1}{2}}$, $\quad f(1)=1$

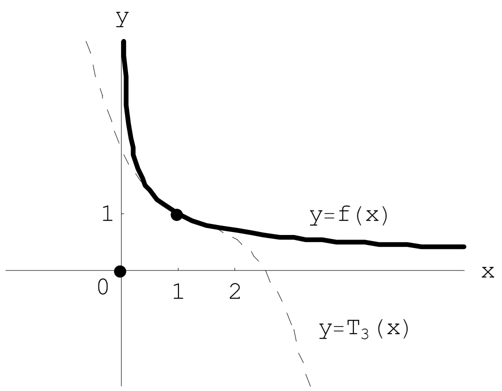
$f'(x)=-\dfrac{1}{2}x^{-\frac{3}{2}}$, $\quad f'(1)=-\dfrac{1}{2}$

$f''(x)=(-\dfrac{1}{2})(-\dfrac{3}{2})x^{-\frac{5}{2}}$, $\quad f''(1)=(-\dfrac{1}{2})(-\dfrac{3}{2})$

$f'''(x)=(-\dfrac{1}{2})(-\dfrac{3}{2})(-\dfrac{5}{2})x^{-\frac{7}{2}}$, $\quad f'''(1)=(-\dfrac{1}{2})(-\dfrac{3}{2})(-\dfrac{5}{2})$

$$f(x)\approx 1+(-\frac{1}{2})(x-a)+\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(x-a)^2+\frac{1}{6}(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(x-a)^3$$

1. aproksimacija: $\dfrac{1}{\sqrt{x}}\approx 1$
2. aproksimacija: $\dfrac{1}{\sqrt{x}}\approx -\dfrac{1}{2}x+\dfrac{3}{2}$
3. aproksimacija: $\dfrac{1}{\sqrt{x}}\approx \dfrac{3}{8}x^2-\dfrac{5}{4}x+\dfrac{15}{8}$
4. aproksimacija: $\dfrac{1}{\sqrt{x}}\approx -\dfrac{5}{16}x^3+\dfrac{21}{16}x^2-\dfrac{35}{16}x+\dfrac{35}{16} = T_3(x)$



□

6. Odabrati m tako da se pravci $\dfrac{x-2}{3}=\dfrac{y-4}{1}=\dfrac{z-2}{1}$ i $\begin{cases} x-z+2=0 \\ y-mz-1=0 \end{cases}$ sijeku i odrediti ravninu koju određuju.

Rješenje:

$p....\quad \frac{x-2}{3}=\frac{y-4}{1}=\frac{z-2}{1},\qquad \vec{p}=\{3,1,1\},\quad P(2,4,2)$

$q....\quad \begin{cases} x-z+2=0 \\ y-mz-1=0 \end{cases},\qquad \vec{n}_1=\{1,0,-1\},\ \vec{n}_2=\{0,1,-m\}$

$\vec{q}=\vec{n}_1\times\vec{n}_2=\{1,m,1\}$

$x=0 \ \Rightarrow\ z=2,\ y=2m+1 \ \Rightarrow\ Q(0,2m+1,2)$

$\overrightarrow{PQ}=\{-2,2m-3,0\}$

Vektori $\vec{q},\vec{p},\overrightarrow{PQ}$ moraju biti komplanarni!

$$\begin{vmatrix} 1 & m & 1 \\ 3 & 1 & 1 \\ -2 & 2m-3 & 0 \end{vmatrix}=0 \ \Rightarrow\ m=2$$

Ravnina:

$$\begin{vmatrix} x-2 & y-4 & z-2 \\ 3 & 1 & 1 \\ 1 & m & 1 \end{vmatrix}=0 \ \Rightarrow\ x+2y-5z=0$$ □