Ispitna zadaća 2:

1. Odrediti sva rješenja jednadžbe $z^4 - (1-i)^{10} = 0$ i nacrtati ih u kompleksnoj ravnini.

2. Zadana je funkcija $y = \sqrt{1 - log_2(x-1)}$.
Odrediti domenu funkcije, domenu inverzne funkcije i derivaciju inverzne funkcije.

3. U polukrugu radijusa $r = 2$ upisati jednakokračan trapez maksimalne površine tako da mu je osnovica jednaka promjeru polukruga.

4. Ispitati funkciju $f(x) = \dfrac{x^3 - 4}{x^2}$ i nacrtati njen graf.

5. Sa i bez L'Hospitalovog pravila izračunati: $\lim\limits_{x\to 0} \dfrac{1 - cos\, x}{x^2}$.

6. Odrediti jednadžbu pravca koji prolazi točkom $T(3,-5,4)$ i presijeca pravce:
$$a \ldots \begin{cases} x = 2z - 1 \\ y = -3z + 4 \end{cases}, \qquad b \ldots \begin{cases} y = -4x + 1 \\ z = x - 2 \end{cases}.$$

Rješenja:

1. Odrediti sva rješenja jednadžbe $z^4 - (1-i)^{10} = 0$ i nacrtati ih u kompleksnoj ravnini.

Rješenje:

$$z^4 - (1-i)^{10} = 0$$
$$(1-i)^{10} = \left[(1-i)^2\right]^5 = \left[1 - 2i + i^2\right]^5 = (-2i)^5 = -2^5 \cdot i^5 = -(2)^5 \cdot i$$
$$(1-i)^{10} = -32i$$
$$z^4 - (-32i) = 0$$
$$z^4 = -32i \qquad \Rightarrow \qquad z = \sqrt[4]{-32i}$$

$\omega = 0 - 32i$ zapisujemo u trigonometrijskom obliku $\omega = |\omega|(cos\,\varphi + i\, sin\,\varphi)$

$$\Rightarrow \qquad |\omega| = 32$$
$$cos\,\varphi = \frac{0}{32} = 0, \quad sin\,\varphi = \frac{-32}{32} = -1, \quad (tg\varphi = -\infty) \Rightarrow \varphi = \frac{3\pi}{2}$$
$$\omega = 32\left(cos\frac{3\pi}{2} + i\, sin\frac{3\pi}{2}\right)$$

$$\left(\sqrt[4]{\omega}\right)_k = \sqrt[4]{|\omega|}\left(cos\frac{\varphi+2k\pi}{4}+i\,sin\frac{\varphi+2k\pi}{4}\right),\ k=0,1,2,3$$

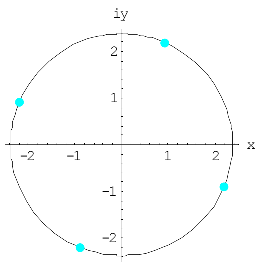
$$\omega_0 = \sqrt[4]{32}\left(cos\frac{\frac{3\pi}{2}}{4}+i\,sin\frac{\frac{3\pi}{2}}{4}\right)=2\sqrt[4]{2}\left(cos\frac{3\pi}{8}+i\,sin\frac{3\pi}{8}\right)$$

$$\omega_1 = \sqrt[4]{32}\left(cos\frac{\frac{3\pi}{2}+2\pi}{4}+i\,sin\frac{\frac{3\pi}{2}+2\pi}{4}\right)=2\sqrt[4]{2}\left(cos\frac{7\pi}{8}+i\,sin\frac{7\pi}{8}\right)$$

$$\omega_2 = \sqrt[4]{32}\left(cos\frac{\frac{3\pi}{2}+4\pi}{4}+i\,sin\frac{\frac{3\pi}{2}+4\pi}{4}\right)=2\sqrt[4]{2}\left(cos\frac{11\pi}{8}+i\,sin\frac{11\pi}{8}\right)$$

$$\omega_3 = \sqrt[4]{32}\left(cos\frac{\frac{3\pi}{2}+6\pi}{4}+i\,sin\frac{\frac{3\pi}{2}+6\pi}{4}\right)=2\sqrt[4]{2}\left(cos\frac{15\pi}{8}+i\,sin\frac{15\pi}{8}\right)$$

Slika:



2. Zadana je funkcija $y=\sqrt{1-log_2(x-1)}$.
Odrediti domenu funkcije, domenu inverzne funkcije i derivaciju inverzne funkcije.

Rješenje:

$$y^2-1+log_2(x-1)=0$$

$$y^2=1-log_2(x-1)$$

$$y=\sqrt{1-log_2(x-1)}$$

Domena:

a) $1-log_2(x-1)\geq 0$ & b) $x-1>0$

$1\geq log_2(x-1)$ $x>1$

$2\geq x-1$

$x\leq 3$

$$\Rightarrow D=\langle 1,3]$$

Inverzna funkcija:

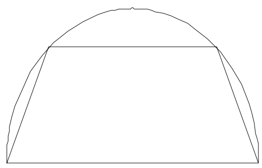
$$f^{-1}(x)=1+2^{\left(1-x^2\right)} \qquad \text{Domena inverzne funkcije:} \quad D=R$$

Derivacija inverzne funkcije:

$$y=1+2^{\left(1-x^2\right)}$$
$$y'=2^{\left(1-x^2\right)}\cdot ln2\cdot\left(1-x^2\right)'$$
$$y'=2^{\left(1-x^2\right)}\cdot ln2\cdot(-2x)=-2x\cdot 2^{\left(1-x^2\right)}\cdot ln2$$

3. U polukrugu radijusa $r = 2$ upisati jednakokračan trapez maksimalne površine tako da mu je osnovica jednaka promjeru polukruga.

Rješenje:



$$P=\frac{(a+c)v}{2}=\left(\frac{a}{2}+\frac{c}{2}\right)\cdot v$$
$$\frac{a}{2}=r=2$$
$$r^2=v^2+\left(\frac{c}{2}\right)^2 \quad\Rightarrow\quad \frac{c}{2}=\sqrt{4-v^2}$$

$$P(v)=v\left(2+\sqrt{4-v^2}\right)=2v+v\sqrt{4-v^2}$$
$$P'(v)=2-\frac{v^2}{\sqrt{4-v^2}}+\sqrt{4-v^2}=0$$

$\Rightarrow$ jedino pozitivno riješenje različito od nule $v_{max}=\sqrt{3}$

$\Rightarrow$ $P_{max}=\sqrt{3}\,\dfrac{4+2\sqrt{4-3}}{2}=3\sqrt{3}$

4. Ispitati funkciju $f(x)=\dfrac{x^3-4}{x^2}$ i nacrtati njen graf.

Rješenje:

a) Domena: $D=R\setminus\{0\}$

b) Parnost, neparnost: Nije parna, nije neparna.

c) Nul točke: $y=0 \Rightarrow x^3-4=0,\ x^3=4,\ x=\sqrt[3]{4}$
Presjeci s osi y: $x=0,\ \ 0\notin D$

d) Ekstremi:

$$y' = \frac{3x^2 \cdot x^2 - 2x\left(x^3 - 4\right)}{x^4} = \frac{x^3 + 8x}{x^3}$$

$$\frac{x^3 + 8}{x^3} = 0 \Rightarrow x^3 = -8 \Rightarrow x = \sqrt[3]{-2} \Rightarrow x = -2, \qquad f(-2) = -3$$

x	$-\infty$	-2		0		$-\infty$
y	↗	M	↘	P	↗	
y'	+		-		+	

P - prekid funkcije
m - minimum finkcije
M - maksimum funkcije
$\Rightarrow \; M(-2,-3)$

e) Infleksija:

$$y'' = \left(1 + 8x^{-3}\right)' = -24x^{-4} = -\frac{24}{x^4} = 0 \Rightarrow \text{ nema točaka infleksije}$$

f) Asimptote:

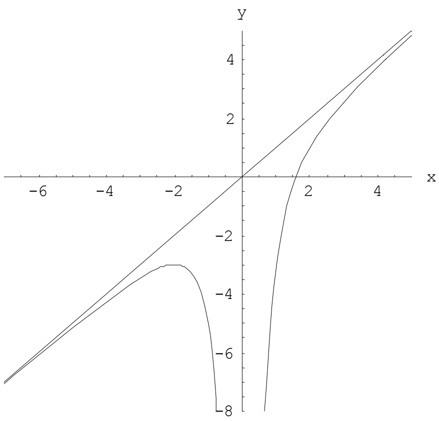
$$\lim_{x\to\pm 0} \frac{x^3 - 4 \;\; / : x^2}{x^2 \;\; / : x^2} = \lim_{x\to 0^+} \frac{x - \frac{4}{x^2}}{1} = -\infty$$

$$\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x^3 - 4}{x^3} = \lim_{x\to\infty} \frac{1 - \frac{4}{x^3}}{1} = 1 = k$$

$$\lim_{x\to\infty} [(f(x) - kx)] = \lim_{x\to\infty} \left(\frac{x^3 - 4}{x^2} - x \right) = \lim_{x\to\infty} \frac{-4}{x^2} = 0 = l$$

Vertikalna asimptota: $x = 0$
Kosa asimptota: $y = x$

g) Graf:



5. Sa i bez L'Hospitalovog pravila izračunati: $\lim\limits_{x\to 0}\frac{1-\cos x}{x^2}$.

Rješenje:

a) Bez L'Hospitalovog pravila:

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2\cdot 4}=\frac{1}{4}\cdot 2\lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2=\frac{1}{2}\cdot 1=\frac{1}{2}$$

b) S L' Hospitalovim pravilom:

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{\sin x}{2x}=\lim_{x\to 0}\frac{\cos x}{2}=\frac{1}{2}$$

6. Odrediti jednadžbu pravca koji prolazi točkom $T(3,-5,4)$ i presijeca pravce:

$$a\ldots\begin{cases}x=2z-1\\ y=-3z+4\end{cases},\qquad b\ldots\begin{cases}y=-4x+1\\ z=x-2\end{cases}.$$

Rješenje:

$$a\cdots\begin{cases}x=2z-1\\ y=-3z+4\end{cases}\Rightarrow\qquad x+1=2z\Rightarrow z=\frac{x+1}{2},\quad y-4=-3z\Rightarrow z=\frac{y-4}{-3}$$

$$a\cdots\ \frac{x+1}{2}=\frac{y-4}{-3}=\frac{z}{1}\qquad \vec{a}=\{2,-3,1\},\quad A(-1,4,0)$$

$$b\cdots\begin{cases}y=-4x+1\\ z=x-2\end{cases}\Rightarrow\qquad -4x=y-1\Rightarrow x=\frac{y-1}{-4},\quad x=z+2$$

$$b\cdots\ \frac{x}{1}=\frac{y-1}{-4}=\frac{z+2}{1}\qquad \vec{b}=\{1,-4,1\},\quad B(0,1,-2)$$

a) Ravnina Π točkom T i pravcem a:

$$\vec{a}=\{2,-3,1\},\quad \overrightarrow{AT}=\{4,-9,4\}$$

$$\begin{vmatrix}x+1 & y-4 & z\\ 2 & -3 & 1\\ 4 & -9 & 4\end{vmatrix}=(x+1)(-12+9)-(y-4)(8-4)+z(-18+12)=$$

$$=-3x-3-4y+16-6z=-3x-4y-6z+13=0\ \cdots\ \Pi$$

b) Probodište *P* pravca *b* i ravnine *Π:*

$$\left.\begin{array}{l} x=t \\ y=-4t+1 \\ z=t-2 \end{array}\right\} \qquad \begin{array}{l} -3(t)-4(-4t+1)-6(t-2)+13=0 \\ -3t+16t-4-6t+12+13=0 \\ 7t=-21 \Rightarrow t=3 \Rightarrow P(-3,13,-5) \end{array}$$

c) Pravac *p* točkama *P* i *T:*

$$\overrightarrow{TP}=\{-3-3,13+5,-5-4\}=\{-6,18,-9\}=3\{-2,6,-3\}$$

$$p \cdots \frac{x-3}{-2}=\frac{y-13}{6}=\frac{z+5}{-3}$$