Ispitna zadaća 4:

1. Odrediti kompleksne brojeve koji zadovoljavaju sljedeće uvjete:

$$\left|\frac{z-12}{z-8i}\right|=\frac{5}{3}, \qquad \left|\frac{z-4}{z-8}\right|=1.$$

2. Odrediti domenu funkcije $f(x)=arcsin\left(\frac{2x}{1+x^2}\right)$ i domenu njene derivacije.

3. Ispitati funkciju $f(x)=\frac{x^3-4}{(x-1)^3}$ i nacrtati njen graf.

4. Pokazati da za male $|x|$ vrijedi približna formula $a\,ch\frac{x}{a}\approx a+\frac{x^2}{2a}$.

5. U lik kojeg zatvara pravac $4y-x=3$ s hiperbolom $y=\frac{3x-1}{x+1}$ upisati trokut maksimalne površine tako da mu se osnovica nalazi na zadanom pravcu. Kolika je to površina?

6. Točkom $M(-3, 1, -2)$ položiti pravac tako da siječe pravce

$$p\ldots\frac{x+5}{2}=\frac{y-3}{-4}=\frac{z+1}{3} \quad \text{i} \quad q\ldots\frac{x-3}{-2}=\frac{y+1}{3}=\frac{z-2}{4}.$$

Rješenja:

1. Odrediti kompleksne brojeve koji zadovoljavaju sljedeće uvjete:

$$\left|\frac{z-12}{z-8i}\right|=\frac{5}{3}, \qquad \left|\frac{z-4}{z-8}\right|=1.$$

Rješenje:

$$z=x+yi$$

$$\frac{z-12}{z-8i}=\frac{x+yi-12}{x+yi-8i}=\frac{(x-12)+yi}{x+(y-8)i}$$

$$\left|\frac{z-12}{z-8i}\right|=\frac{|(x-12)+yi|}{|x+(y-8i)|}=\frac{\sqrt{(x-12)^2+y^2}}{\sqrt{x^2+(y-8)^2}}=\frac{5}{3}\Rightarrow$$

$$3\sqrt{(x-12)^2+y^2}=5\sqrt{x^2+(y-8)^2} \quad /^2$$

$$9\left(x^2-24x+144+y^2\right)=25\left(x^2+y^2-16y+64\right)$$

$$\vdots$$

$$2x^2+2y^2+27x-50y-27\cdot 6+25\cdot 8=0$$

$$\left|\frac{z-4}{z-8}\right|=\frac{|x+yi-4|}{|x+yi-8|}=\frac{\sqrt{(x-4)^2+y^2}}{\sqrt{(x-8)^2+y^2}}=1$$

$$\sqrt{(x-4)^2+y^2}=\sqrt{(x-8)^2+y^2} \quad /^2$$

$$(x-4)^2+y^2=(x-8)^2+y^2$$

$$\vdots$$

$$x=6$$

$$x=6 \quad \& \quad 2x^2+2y^2+27x-50y-27\cdot 6+25\cdot 8=0 \quad \Rightarrow \quad \begin{cases} z_1=6+8i \\ z_2=6+17i \end{cases} \square$$

2. Odrediti domenu funkcije $f(x)=arcsin\left(\frac{2x}{1+x^2}\right)$ i domenu njene derivacije.

Rješenje:

$$1+x^2\neq 0,\ \forall x\in R$$

$$1+x^2\geq 1,\ \forall x\in R$$

$$-1\leq\frac{2x}{1+x^2}\leq 1 \Leftrightarrow \qquad -1\leq\frac{2x}{1+x^2} \quad \& \quad \frac{2x}{1+x^2}\leq 1$$

a) & b)

a)	b)
$-1\leq\frac{2x}{1+x^2}$	$\frac{2x}{1+x^2}\leq 1$
$-1-x^2\leq 2x$	$2x\leq 1+x^2$
$-x^2-2x-1\leq 0$	$-x^2+2x-1\leq 0$
$x^2+2x+1\geq 0$	$x^2-2x+1\geq 0$
$(x+1)^2\geq 0$	$(x-1)^2\geq 0$
$x\in R$	$x\in R$

Domena od $f(x)$: $\quad D=R$

Derivacija:

$$f'(x)=\frac{1}{\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}}\cdot\left(\frac{2x}{1+x^2}\right)'$$

$$\vdots$$

$$f'(x)=\frac{2\left(1-x^2\right)}{\left|1-x^2\right|\left(1+x^2\right)}$$

a)

$$-1+x^2\neq 0$$
$$x^2\neq 1$$
$$x\neq\pm 1$$

b)

$$1+x^2\geq 1,\ \ \forall x\in R$$

Domena od $f'(x)$: $D=R\setminus\{-1,1\}$

3. Ispitati funkciju $f(x)=\frac{x^3-4}{(x-1)^3}$ i nacrtati njen graf.

Rješenje:

a) Domena: $D=R\setminus\{1\}$

b) Parnost, neparnost: Nije parna, nije neparna (domena nije simetrična prema ishodištu).

c) Nul točke: $y=0\Rightarrow x^3-4=0,\ x=\sqrt[3]{4}$

Presjeci s osi y: $x=0\ \Rightarrow\ f(0)=4$

d) Ekstremi:

$$f'(x)=\frac{3x^2(x-1)^3-3(x-1)^2\cdot\left(x^3-4\right)}{(x-1)^6}=\frac{3x^2(x-1)-3\left(x^3-4\right)}{(x-1)^4}$$

$$f'(x)=\frac{3\left(4-x^2\right)}{(x-1)^4}=0$$

$$\left(4-x^2\right)=0$$

$$(2-x)(2+x)=0\quad\Rightarrow\quad x_1=2,\ \ x_2=-2$$

x	-∞	-2		1		2		+∞
y	↘	m	↗	P	↗	M	↘	
y'	-		+		+		-	

P - prekid funkcije
m - minimum finkcije
M - maksimum funkcije

$$x_1 = 2 \quad \Rightarrow f(x_1) = 4 \qquad \Rightarrow \quad M(2,4)$$

$$x_2 = -2 \Rightarrow f(x_2) = \frac{12}{27} \qquad \Rightarrow \quad m\left(-2, \frac{12}{27}\right)$$

e) Infleksija:

$$f''(x) = \frac{3(-2x)(x-1)^4 - 4(x-1)^3 \cdot 3(4-x^2)}{(x-1)^8} = \cdots =$$

$$f''(x) = \frac{6x^2 + 6x - 48}{(x-1)^5} = 0 \qquad \Rightarrow \qquad x_{I1} = \frac{-1+\sqrt{33}}{2}, \quad x_{I2} = \frac{-1-\sqrt{33}}{2}$$

Točke infleksije su:

$$I_1 = \left(\frac{-1+\sqrt{33}}{2}, f''(x_{I1})\right)$$

$$I_2 = \left(\frac{-1-\sqrt{33}}{2}, f''(x_{I2})\right)$$

f) Asimptote:

$$\lim_{x\to 1^+} \frac{x^3 - 4}{(x-1)^3} = \frac{-3}{0^+} = -\infty$$

$$\lim_{x\to 1^-} \frac{x^3 - 4}{(x-1)^3} = \frac{-3}{0^-} = +\infty$$

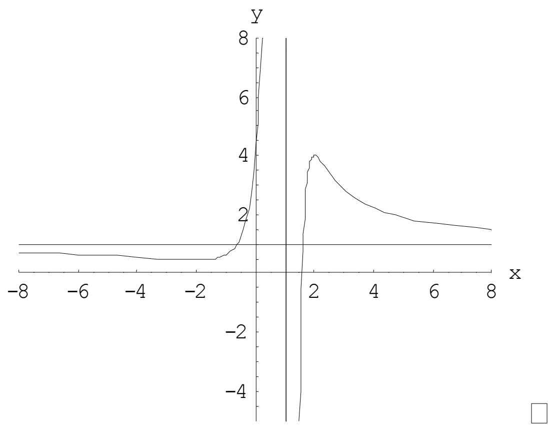
Vertikalna asimptota: $x = 1$

$$\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x^3 - 4 \quad / : x^3}{(x-1)^3 \cdot x \quad / : x^3} = \lim_{x\to\infty} \frac{1 - \frac{4}{x^3}}{\left(1 - \frac{1}{x}\right)^3 \cdot x} = 0 = k$$

$$\lim_{x\to\infty} [f(x) - kx] = \lim_{x\to\infty} \frac{x^3 - 4}{(x-1)^3} = \lim_{x\to\infty} \frac{1 - \frac{4}{x^3}}{\left(1 - \frac{1}{x}\right)^3} = 1 = l$$

Horizontalna asimptota: $y = 1$

g) Graf:



4. Pokazati da za male $|x|$ vrijedi približna formula

$$a\,ch\frac{x}{a} \approx a + \frac{x^2}{2a}.$$

Rješenje:

Taylorov razvoj funkcije $y = f(x)$ u red u okolini nule:

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{IV}(0) + \cdots$$

Za funkciju $f(x) = a\,ch\frac{x}{a}$ izračunamo redom:

$$f(x) = a\,ch\frac{x}{a}, \qquad f(0) = a$$

$$f'(x) = sh\frac{x}{a}, \qquad f'(0) = 0$$

$$f''(x) = \frac{1}{a}ch\frac{x}{a}, \qquad f''(0) = \frac{1}{a}$$

$$f'''(x) = \frac{1}{a^2}sh\frac{x}{a}, \qquad f'''(0) = 0$$

$$f^{IV}(x) = \frac{1}{a^3}ch\frac{x}{a}, \qquad f^{IV}(0) = \frac{1}{a^3}, \ \ldots$$

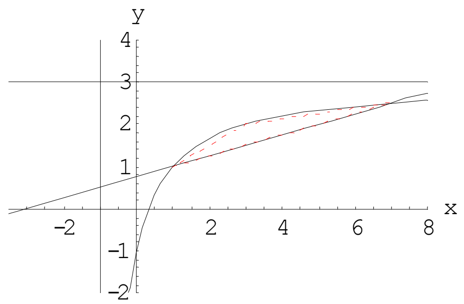
$$\Rightarrow \quad a\,ch\frac{x}{a} = a + \frac{x^2}{2a} + \frac{x^4}{24a^3} + \frac{x^6}{720a^5} + \cdots$$

Iz toga se vidi da se za male vrijednosti $|x|$ može uzeti da formula **vrijedi,** jer se viši članovi mogu zanemariti budući da poprimaju sve manje i manje vrijednosti.

5. U lik kojeg zatvara pravac $4y-x=3$ s hiperbolom $y=\frac{3x-1}{x+1}$ upisati trokut maksimalne površine tako da mu se osnovica nalazi na zadanom pravcu. Kolika je to površina?

Riješenje:

Presjek hiperbole i pravca:



$$\begin{aligned} y&=\frac{3x-1}{x+1}\\ y&=\frac{1}{4}x+\frac{3}{4}\end{aligned}$$

$$\frac{3x-1}{x+1}=\frac{1}{4}x+\frac{3}{4} \Rightarrow \begin{array}{ll} x_1=1\Rightarrow y_1=1 & T_1(1,1) \\ x_2=7\Rightarrow y_2=\frac{5}{2} & T_2\left(7,\frac{5}{2}\right)\end{array}$$

Udaljenost točke (x_0,y_0) na hiperboli od pravca $Ax+By+C=0$ izračunamo po formuli:

$$d=\frac{Ax_0+By_0+C}{\pm\sqrt{A^2+B^2}}.$$

Pravac: $-x+4y-3=0$

Točka na hiperboli: $(x_0,y_0)=\left(x_{0,}\ \frac{3x_0-1}{x_0+1}\right)$

$$d=\frac{-x_0+4y_0-3}{+\sqrt{1+16}}=\frac{-x_0+4\left(\frac{3x_0-1}{x_0+1}\right)-3}{\sqrt{17}}=\frac{-x_0^2+8x_0-7}{\sqrt{17}(x_0+1)}$$

Površina trokuta: $P=\frac{B\cdot v}{2}$

B - udaljenost točaka $T_1\,T_2$

$$B=\sqrt{(7-1)^2+\left(\frac{5}{2}-1\right)^2}=\frac{\sqrt{153}}{2}$$

$$P(x_0)=\frac{\sqrt{153}}{2}\left(\frac{-x_0^2+8x_0-7}{\sqrt{17}(x_0+1)}\right)=\frac{3}{2}\left(\frac{-x_0^2+8x_0-7}{(x_0+1)}\right)$$

$$P'(x_0)=\frac{3}{2}\left(\frac{-x_0^2-2x_0+15}{(x_0+1)^2}\right)=0$$

$$x_0^2+2x_0-15=0 \Rightarrow \begin{matrix}(x_0)_1=-5\\(x_0)_2=3\Rightarrow y=2\Rightarrow(3,2)\end{matrix}$$

$$P=\frac{3}{2}\left(\frac{-9+24-7}{4}\right)=3$$

6. Točkom $M(-3,1,-2)$ položiti pravac tako da siječe pravce

$$p\ldots\frac{x+5}{2}=\frac{y-3}{-4}=\frac{z+1}{3} \quad \text{i} \quad q\ldots\frac{x-3}{-2}=\frac{y+1}{3}=\frac{z-2}{4}.$$

Rješenje:

a) Ravnina Π točkom M i pravcem $\vec{p}$:

$$\vec{p}=\{2,-4,3\}$$
$$P=(-5,3,-1)$$
$$\overrightarrow{MP}=\{-2,2,1\}$$

$$\vec{p}\times\overrightarrow{MP}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & -4 & 3\\ -2 & 2 & 1\end{vmatrix}=-10\vec{i}-8\vec{j}-4\vec{k} \quad\Rightarrow\quad \vec{n}=\{5,4,2\}$$

$$\Pi\ldots 5(x+3)+4(y-1)+2(z+2)=0$$
$$\Pi\ldots 5x+4y+2z+15=0$$

b) Probodište pravca q i ravnine Π:

$$q\begin{cases}x=-2t+3\\ y=3t-1\\ z=4t+2\end{cases} \quad \& \quad \Pi\ldots 5x+4y+2z+15=0 \Rightarrow t=-3$$

$$N\begin{cases}x=6+3=9\\ y=-9-1=-10\\ z=-12+2=-10\end{cases} \quad\Rightarrow\quad N(9,-10,-10)$$

c) Transverzala t točkama M i N:

$$\vec{t}=\overrightarrow{MN}=\{12,-11,-8\}$$

$$t\ldots\quad\frac{x+3}{12}=\frac{y-1}{-11}=\frac{z+2}{-8}$$