Ispitna zadaća 6:

1. Naći član binomnog razvoja $\left(3xy^2+z^2\right)^7$ koji sadrži y^6.

2. Odrediti domenu funkcije $g(x)=\sqrt{ln\left(3+x-x^2\right)}$ i napisati jednadžbu tangente na njen graf u točki $x_0=0$.

3. Analizirati funkciju $f(x)=\sqrt[3]{x^3-x^2-x+1}$ i skicirati njen graf.

4. U polukrug polumjera *R* upisati trapez maksimalne površine.

5. Dani su vektori $\vec{a}=\{1,1,1\}$, $\vec{b}=\{3,2,1\}$, $\vec{c}=\{1,2,3\}$, $\vec{d}=\{4,0,1\}$, $\vec{e}=\{2,1,2\}$.
Odrediti skalare λ i μ tako da vektor $\vec{a}+\lambda\vec{b}+\mu\vec{c}$ bude okomit na vektorima $\vec{d}$ i $\vec{e}$.

6. Točkom $T(-5,16,12)$ položiti ravnine od kojih jedna sadrži os *x*, a druga sadrži os *y*. Odrediti kut između tih ravnina.

Rješenja:

1. Naći član binomnog razvoja $\left(3xy^2+z^2\right)^7$ koji sadrži y^6.

Rješenje:

$$(a+b)^n=a^n+\binom{n}{1}a^{n-1}b+\binom{n}{2}a^{n-2}b^2+\cdots+\binom{n}{n-2}a^2b^{n-2}+\binom{n}{n-1}ab^{n-1}+b^n$$

$$(a+b)^n=\sum_{k=n}^{0}\binom{n}{n-k}a^kb^{n-k}=\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k.$$

$$\left(3xy^2+z^2\right)^7=\sum_{k=0}^{7}\binom{7}{k}\left(3xy^2\right)^{7-k}\cdot\left(z^2\right)^k$$

$$3^{7-k}\cdot x^{7-k}\cdot y^{2(7-k)}\cdot z^{2k}\quad\Rightarrow\quad y^{14-2k}=y^6\quad\Rightarrow\quad 14-2k=6\quad\Rightarrow\quad k=4$$

Traženi član je:

$$\binom{7}{4}3^{7-4}\cdot x^{7-4}\cdot y^{2(7-4)}\cdot z^{2\cdot 4}=3^3\,\frac{7\cdot 6\cdot 5\cdot 4}{1\cdot 2\cdot 3\cdot 4}\cdot x^3y^6z^8=27\cdot 35x^3y^6z^8=945x^3y^6z^8\ \square$$

2. Odrediti domenu funkcije $g(x)=\sqrt{ln(3+x-x^2)}$ i napisati jednadžbu tangente na njen graf u točki $x_0=0.$

Rješenje:

Domena:

a) & b)

a)

$$ln(3+x-x^2)\geq 0$$
$$3+x-x^2\geq 1$$
$$-x^2+x+2\geq 0$$
$$-x^2+x+2=0 \Rightarrow x_1=2 \ \& \ x_2=-1$$
$$x\in[-1,2]$$

b)

$$3+x-x^2>0$$
$$-x^2+x+3=0 \Rightarrow$$
$$x_1=\frac{1-\sqrt{13}}{2} \ \& \ x_2=\frac{1+\sqrt{13}}{2}$$
$$x\in\left[\frac{1-\sqrt{13}}{2},\frac{1+\sqrt{13}}{2}\right]$$

$$\text{a)} \cap \text{b)} \Rightarrow D=[-1,2]$$

Jednadžba tangente:

$$g'(x)=\frac{1}{2\sqrt{ln(3+x-x^2)}}\cdot\frac{1}{3+x-x^2}\cdot(-2x+1)$$
$$x_0=0 \Rightarrow g(x_0)=\sqrt{ln3} \Rightarrow g'(x_0)=\frac{1}{6\sqrt{ln3}}$$
$$y-y_0=g'(0)(x-x_0) \Rightarrow y=\frac{1}{6\sqrt{ln3}}x+\sqrt{ln3}$$ □

3. Analizirati funkciju $f(x)=\sqrt[3]{x^3-x^2-x+1}$ i skicirati njen graf.

Rješenje:

a) Domena: $D=R$

b) Nul točke: $y=0 \Rightarrow$

$$x^3-x^2-x+1=x^2(x-1)-(x-1)=(x-1)(x^2-1)=0 \Rightarrow$$
$$x=\pm 1$$

Presjeci s osi y: $x=0 \Rightarrow y=1$

b) Parnost, neparnost: Nije parna, nije neparna.

c) Ekstremi:

$$f'(x)=\frac{1}{3}(x^3-x^2-x+1)^{-\frac{2}{3}}\cdot(3x^2-2x-1)=\frac{3x^2-2x-1}{3\sqrt[3]{(x^3-x^2-x+1)^2}}=0$$

$3x^2 - 2x - 1 = 0 \quad \Rightarrow \quad x_1 = -\frac{1}{3} \quad \& \quad x_2 = 1$

$x^3 - x^2 - x + 1 = 0 \quad \Rightarrow \quad x = \pm 1 \quad \Rightarrow$ Tangenta paralelna s osi y.

x	$-\infty$	-1	$-1/3$	1	$-\infty$
y	↗	↗	M ↘	m ↗	
y'	+	+	-	+	

m - minimum finkcije
M - maksimum funkcije

$$x_1 = -\frac{1}{3} \quad \Rightarrow \quad f(x_1) = \frac{2\sqrt[3]{4}}{3} \quad \Rightarrow \quad M\left(-\frac{1}{3}, \frac{2\sqrt[3]{4}}{3}\right)$$

$$x_2 = 1 \quad \Rightarrow \quad f(x_2) = 0 \quad \Rightarrow \quad m(1,0)$$

d) Asimptote:

$$\lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \sqrt[3]{\frac{x^3 - x^2 - x + 1}{x^3}} = \lim_{x\to\pm\infty} \sqrt[3]{1 - \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3}} = 1 = k$$

$$\lim_{x\to\pm\infty} (f(x) - k \cdot x) = \lim_{x\to\pm\infty} \left(\sqrt[3]{x^3 - x^2 - x + 1} - x\right) =$$

$$= \lim_{x\to\pm\infty} \left(\sqrt[3]{x^3 - x^2 - x + 1} - x\right) \frac{\left(\left(\sqrt[3]{x^3 - x^2 - x + 1}\right)^2 + x\sqrt[3]{x^3 - x^2 - x + 1} + x^2\right)}{\left(\left(\sqrt[3]{x^3 - x^2 - x + 1}\right)^2 + x\sqrt[3]{x^3 - x^2 - x + 1} + x^2\right)} =$$

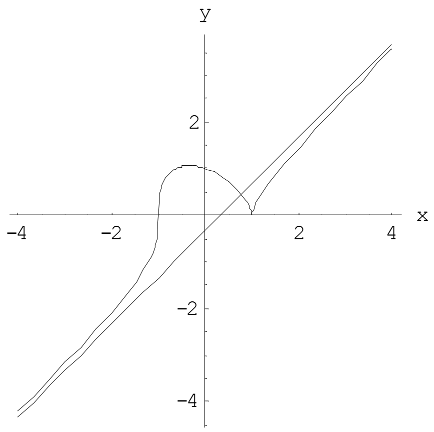
$$= \lim_{x\to\pm\infty} \frac{x^3 - x^2 - x + 1 - x^3 \qquad /:x^2}{\left(\sqrt[3]{x^3 - x^2 - x + 1}\right)^2 + x\sqrt[3]{x^3 - x^2 - x + 1} + x^2 \qquad /:x^2} =$$

$$= \lim_{x\to\pm\infty} \frac{-1 - \frac{1}{x} + \frac{1}{x^2}}{\sqrt[3]{\left(1 - \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3}\right)^2} + \sqrt[3]{1 - \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3}} + 1} = -\frac{1}{3} = l$$

Kosa asimptota: $\quad y = x - \frac{1}{3}$

Nema horizontalnih asimptota: $\quad \lim_{x\to\pm\infty} f(x) = \pm\infty$

e) Graf:



□

4. U polukrug polumjera R upisati trapez maksimalne površine.

Rješenje:

Površina trapeza: $P_{TR} = \dfrac{(a+c)\cdot v}{2}$.

Osnovica $a = 2R$; dužina $x = \dfrac{a-c}{2}$, gornja osnovica $c = 2R - 2x = 2(R-x)$.

Visina:

$$v^2 = R^2 - (R-x)^2$$
$$v^2 = 2Rx - x^2$$
$$v = \sqrt{x(2R-x)}.$$

$$P_{TR}(x) = \frac{(2R+2R-2x)\sqrt{x(2R-x)}}{2} = (2R-x)\sqrt{x(2R-x)}$$

$$P'_{TR}(x) = -\sqrt{x(2R-x)} + (2R-x)\frac{1}{2\sqrt{x(2R-x)}}\cdot(2R-2x)$$

$$P'_{TR}(x) = \frac{2x^2 - 5Rx + 2R^2}{\sqrt{x(2R-x)}} = 0$$

$$2x^2 - 5Rx + 2R^2 = 0$$
$$x_{1,2} = \frac{5R \pm 3R}{4} \quad \Rightarrow \quad x_1 = \frac{R}{2}, \quad x_2 = 2R > R \ \ \textit{otpada}$$

$$v^2 = 2R\cdot\frac{R}{2} - \frac{R^2}{4}$$
$$v = \sqrt{R^2\left(1-\frac{1}{4}\right)} = R\sqrt{\frac{3}{4}}$$

$$\alpha = 60^\circ$$
$$c = R$$

$$P = (2R+R)\cdot\frac{R\sqrt{3}}{4} = \frac{3\sqrt{3}}{4}R^2$$ □

5. Dani su vektori $\vec{a}=\{1,1,1\}$, $\vec{b}=\{3,2,1\}$, $\vec{c}=\{1,2,3\}$, $\vec{d}=\{4,0,1\}$, $\vec{e}=\{2,1,2\}$.
Odrediti skalare λ i μ tako da vektor $\vec{a}+\lambda\vec{b}+\mu\vec{c}$ bude okomit na vektorima $\vec{d}$ i $\vec{e}$.

Rješenje:

Neka je vektor $\vec{m}=\vec{a}+\lambda\vec{b}+\mu\vec{c}$.

$$\vec{m}\perp\vec{d} \quad \& \quad \vec{m}\perp\vec{e} \quad \Rightarrow \quad \vec{m}\cdot\vec{d}=0 \quad \& \quad \vec{m}\cdot\vec{e}=0$$

$$\left(\vec{a}+\lambda\vec{b}+\mu\vec{c}\right)\cdot\vec{d}=0 \qquad \left(\vec{a}+\lambda\vec{b}+\mu\vec{c}\right)\cdot\vec{e}=0$$

$$\vec{a}\vec{d}+\lambda\vec{b}\vec{d}+\mu\vec{c}\vec{d}=0 \qquad \vec{a}\vec{e}+\lambda\vec{b}\vec{e}+\mu\vec{c}\vec{e}=0$$

$$\Rightarrow 13\lambda+7\mu+5=0 \qquad \Rightarrow 10\lambda+10\mu+5=0$$

$$\begin{aligned}10\lambda+10\mu&=-5\\13\lambda+7\mu&=-5\end{aligned} \quad \Rightarrow \quad \begin{aligned}\lambda&=-\frac{1}{4}\\ \mu&=-\frac{1}{4}\end{aligned} \square$$

6. Točkom $T(-5,16,12)$ položiti ravnine od kojih jedna sadrži os x, a druga sadrži os y. Odrediti kut između tih ravnina.

Rješenje:

Vektori $\overrightarrow{OT}$ & $\vec{i}$ određuju ravninu $\Sigma_x = \begin{vmatrix} x & y & z \\ 1 & 0 & 0 \\ -5 & 16 & 12 \end{vmatrix} = 0$, odnosno

vektori $\overrightarrow{OT}$ & $\vec{j}$ određuju ravninu $\Sigma_y = \begin{vmatrix} x & y & z \\ 0 & 1 & 0 \\ -5 & 16 & 12 \end{vmatrix} = 0$.

Njihove jednadžbe i vektori normala su:

$$\begin{aligned} &x\cdot 0-y\cdot 12+z\cdot 16=0 \\ &-3y+4z=0 \\ &\overrightarrow{n_x}=\{0,-3,4\} \end{aligned} \qquad \begin{aligned} &x\cdot 12-y\cdot 0+z\cdot 5=0 \\ &12x+5z=0 \\ &\overrightarrow{n_y}=\{12,0,5\} \end{aligned}$$

Kut između ravnina, tj. kut između vektora normala dobije se iz:

$$\cos\alpha=\frac{\overrightarrow{n_x}\cdot\overrightarrow{n_y}}{\left|\overrightarrow{n_x}\right|\cdot\left|\overrightarrow{n_y}\right|} \quad \Rightarrow \quad \cos\alpha=\frac{\{0,-3,4\}\cdot\{12,0,5\}}{\sqrt{144+25}\cdot\sqrt{9+16}}=\frac{4}{13}\Rightarrow$$

$$\alpha=72^\circ 04' 47'' \square$$