Ispitna zadaća 8:

1. Riješiti jednadžbu $2z^3 - 1 - i = 0$.

2. Odrediti domenu i skicirati graf funkcije $f(x) = \frac{x+2}{x-1}$, te napisati jednadžbu tangente iz točke $T(1,-\frac{3}{2})$ na tu krivulju.

3. Odrediti domenu, presjek s koordinatnim osima, ekstreme, asimptote i skicirati graf funkcije $y = \frac{x^3}{x^2-1}$.

4. Napisati polinom 3. stupnja u točki $x_0 = 0$ za funkciju $f(x) = arcsin\, x$.

5. Odrediti funkciju koja je inverzna funkciji $f(x) = \sqrt[3]{x + \sqrt{1+x^2}} + \sqrt[3]{x - \sqrt{1+x^2}}$.

6. Odrediti jednadžbu pravca koji leži u ravnini $3x - 2y - 2z = 1$, prolazi točkom $T(1,0,1)$ i okomit je na pravac $3x = 2y = z$.

Rješenja:

1. Riješiti jednadžbu $2z^3 - 1 - i = 0$.

Rješenje:

$$z = \sqrt[3]{\frac{1}{2} + \frac{1}{2}i},$$

$$u = \frac{1}{2} + \frac{1}{2}i, \quad |u| = \frac{\sqrt{2}}{2} \quad \Rightarrow \quad u = \frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$$

$$cos\,\varphi = \frac{\sqrt{2}}{2}, \; sin\,\varphi = \frac{\sqrt{2}}{2} \quad \Rightarrow \quad \varphi = \frac{\pi}{4}$$

$$z_k = \sqrt[3]{\frac{\sqrt{2}}{2}}(\cos\frac{\frac{\pi}{4} + k2\pi}{3} + i\sin\frac{\frac{\pi}{4} + k2\pi}{3}), \quad \text{k = 0, 1, 2}$$

$$z_0 = \sqrt[3]{\frac{\sqrt{2}}{2}}\left(cos\,\frac{\pi}{12} + i\,sin\,\frac{\pi}{12}\right)$$

$$z_1 = \sqrt[3]{\frac{\sqrt{2}}{2}}\left(cos\,\frac{9\pi}{12} + i\,sin\,\frac{9\pi}{12}\right)$$

$$z_2 = \sqrt[3]{\frac{\sqrt{2}}{2}}\left(\cos\frac{17\pi}{12} + i\sin\frac{\pi}{12}\right) \square$$

2. Odrediti domenu i skicirati graf funkcije $f(x) = \dfrac{x+2}{x-1}$, te napisati jednadžbu tangente iz točke $T(1,-\frac{3}{2})$ na tu krivulju.

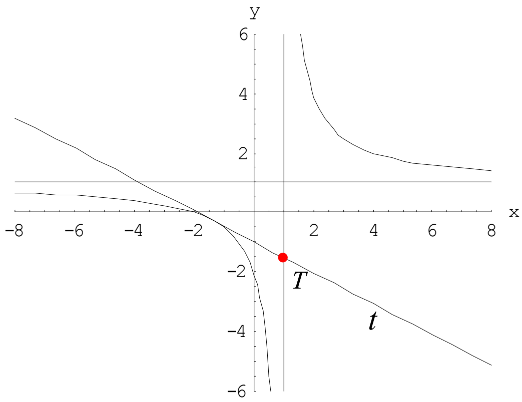
Rješenje:

Domena: D $= R \setminus \{1\}$

Asimptote:

Vertikalna asimptota: $\lim\limits_{x\to 1^+} \dfrac{x+2}{x-1} = \infty, \quad \lim\limits_{x\to 1^-} \dfrac{x+2}{x-1} = -\infty \quad \Rightarrow \quad x = 1$

Horizontalna asimptota: $\lim\limits_{x\to \infty} \dfrac{x+2}{x-1} = 1 \quad \Rightarrow \quad y = 1$



Jednadžba tangente: $y - y_0 = f'(x_0)(x - x_0)$

$$f(x) = \frac{x+2}{x-1}, \quad f'(x) = -\frac{3}{(x-1)^2}$$

$$y - \frac{x_0+2}{x_0-1} = -\frac{3}{(x_0-1)^2}(x - x_0),$$

Točkom $T(1,-\frac{3}{2})$: $\quad -\dfrac{3}{2} - \dfrac{x_0+2}{x_0-1} = -\dfrac{3}{(x_0-1)^2}(1 - x_0)$

$$\Rightarrow \quad x_0 = -\frac{7}{5}, \quad y_0 = \frac{x_0+2}{x_0-1} = -\frac{1}{4}$$

Dirالište tangente: $(x_0, y_0) = \left(-\dfrac{7}{5}, -\dfrac{1}{4}\right)$

Tangenta: $y = -\dfrac{25}{48}x - \dfrac{47}{48} \square$

3. Odrediti domenu, presjek s koordinatnim osima, ekstreme, asimptote i skicirati graf funkcije $y = \frac{x^3}{x^2 - 1}$.

Rješenje:

a) Domena: $x^2 - 1 \neq 0 \Rightarrow D = R \setminus \{-1,1\}$

b) Parnost, neparnost: $f(x) = f(-x)$ $\Rightarrow$ funkcija je neparna

c) Nul točke: $y = 0 \Rightarrow$ točka $x = 0$ je trostruka
Presjeci s osi y: $x = 0 \Rightarrow y = 0$

d) Ekstremi:

$$f'(x) = -\frac{2x^4}{(-1+x^2)^2} + \frac{3x^2}{-1+x^2}, \quad f'(x) = 0 \quad \Rightarrow \quad x_{1,2} = 0, x_3 = -\sqrt{3}, x_4 = \sqrt{3}$$

x	-∞		$-\sqrt{3}$		-1		0		1		$-\sqrt{3}$		+∞
y		↗	M	↘	P	↘		↘	P	↘	m	↗	
y'		+		-		-		-		-		+	

P - prekid funkcije
m - minimum finkcije
M - maksimum funkcije

$$\Rightarrow \quad M\left(-\sqrt{3}, -\frac{3\sqrt{3}}{2}\right), \quad m\left(\sqrt{3}, \frac{3\sqrt{3}}{2}\right)$$

e) Asimptote:

$$\lim_{x \to 1^-} \frac{x^3}{x^2 - 1} = \infty, \qquad \lim_{x \to 1^+} \frac{x^3}{x^2 - 1} = -\infty$$

Budući da je funkcija neparna ne moramo računati $\lim_{x \to -1^-} f(x)$ i $\lim_{x \to -1^+} f(x)$.

Vertikalne asimptote: $x = -1, \ x = 1$

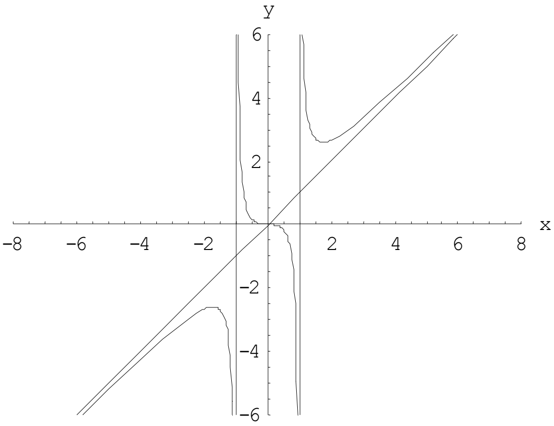
$$\lim_{x \to \infty} \frac{x^3}{x^2 - 1} = \infty \qquad \Rightarrow \quad \text{nema horizontalne asimptote}$$

$$\lim_{x \to \pm\infty} \frac{f(x)}{x} = \lim_{x \to \pm\infty} \left(\frac{x^3}{x^3 - x} \right) = 1 = k$$

$$\lim_{x \to \pm\infty} (f(x) - ax) = \lim_{x \to \pm\infty} \left(\frac{x^3}{x^2 - 1} - x \right) = 0 = l$$

Kosa asimptota: $y = x$

f) Graf:



□

4. Napisati polinom 3. stupnja u točki $x_0 = 0$ za funkciju $f(x) = arcsin\, x$.

Rješenje:

$f(x) = arcsin\, x.$

$f'(x) = (1-x^2)^{-\frac{1}{2}},\ f'(0) = 1$

$f''(x) = x(1-x^2)^{-\frac{3}{2}},\ f''(0) = 0$

$f'''(x) = 3x^2(1-x^2)^{-\frac{5}{2}} + (1-x^2)^{-\frac{3}{2}},\ f'''(0) = 1$

$arcsinx \approx x + \dfrac{x^3}{6}$ □

5. Odrediti funkciju koja je inverzna funkciji $f(x) = \sqrt[3]{x+\sqrt{1+x^2}} + \sqrt[3]{x-\sqrt{1+x^2}}$.

Rješenje:

Domena funkcije: $1 + x^2 > 0 \ \Rightarrow\ D = R$

Monotonost funkcije:

$$f'(x) = \frac{1}{3}\left(\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}\,\left(x+\sqrt{1+x^2}\right)^{2/3}}\right) + \frac{1}{3}\left(\frac{-x+\sqrt{1+x^2}}{\sqrt{1+x^2}\,\left(x-\sqrt{1+x^2}\right)^{2/3}}\right) =$$

$$= \frac{1}{3}\,\frac{\left(x-\sqrt{1+x^2}\right)^{2/3}\left(x+\sqrt{1+x^2}\right) + \left(x+\sqrt{1+x^2}\right)^{2/3}\left(-x+\sqrt{1+x^2}\right)}{\sqrt{1+x^2}}$$

Nejednakost $|x| < \left|\sqrt{1+x^2}\right| \ \Rightarrow\ f'(x) > 0 \ \Rightarrow\ f(x)$ strogo raste za $\forall x \in D$

Inverzna funkcija:

Zamijenimo varijable x i y i odredimo $y = f^{-1}(x)$.

$$x = \sqrt[3]{y + \sqrt{1 + y^2}} + \sqrt[3]{y - \sqrt{1 + y^2}} \quad \Big|^3$$

$$\vdots$$

$$x^3 = 2y - 3\left(\sqrt[3]{y + \sqrt{1 + y^2}} + \sqrt[3]{y - \sqrt{1 + y^2}}\right) = 2y - 3x$$

$$\Rightarrow \quad y = \frac{1}{2}x^3 + \frac{3}{2}x \; \Box$$

6. Odrediti jednadžbu pravca koji leži u ravnini $3x - 2y - 2z = 1$, prolazi točkom $T(1,0,1)$ i okomit je na pravac $3x = 2y = z$.

Rješenje:

$$\Pi \; \ldots \; 3x - 2y - 2z = 1$$

$$p \;\ldots\ldots\; 3x = 2y = z \quad \Rightarrow \quad \frac{x}{2} = \frac{y}{3} = \frac{z}{6} \quad \Rightarrow \quad \vec{p} = \{2,3,6\}$$

Ravnina Σ točkom T okomito na pravac p:

$$\Sigma \;\ldots\ldots\; \begin{array}{l} 2(x-1) + 3(y-0) + 6(z-1) = 0 \\ 2x + 3y + 6z - 8 = 0 \end{array}$$

Traženi pravac $q = \Pi \cap \Sigma$:

$$\vec{n}_1(\Pi) = \{3,-2,-2\}, \qquad \vec{n}_2(\Sigma) = \{2,3,6\}$$

$$\vec{q} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -2 & -2 \\ 2 & 3 & 6 \end{vmatrix} = -6\vec{i} - 22\vec{j} + 13\vec{k}$$

$$T \in \Pi \Rightarrow \qquad q \ldots\ldots \frac{x-1}{6} = \frac{y}{22} = \frac{z-1}{-13} \; \Box$$